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ABSTRACT

This document contains the proceedings of the annual meeting of the Canadian Mathematics Education Study Group. Papers include: (1) "What Does It Really Mean To Teach Mathematics through Inquiry?" (Raffaella Borasi); (2) "The High School Math Curriculum" (Peter Taylor); (3) "Triple Embodiment: Studies of Mathematical Understanding-in-Inter-action in My Work and in the Work of CMESG/GCEDM" (Thomas E. Kieren); (4) "Awareness and Expression of Generality in Teaching Mathematics" (Louis Charbonneau and John Mason); (5) "Communicating Mathematics" (Douglas Franks and Susan Pirie); (6) "The Crisis in School Mathematics Content" (Malgorzata Dubiel and David Reid); (7) "Abstract Algebra: A Problems-centered and Historically Focused Approach" (Israel Kleiner); (8) "Algebraic Understanding" (Lesley Lee); (9) "Students' Explanations in Linear Algebra" (Tommy Dreyfus); (10) "Mathematics Teaching--How It Could Be Done" (George Kondor); (11) "Mathematics Teachers' Needs in Dynamic Geometric Computer Environments: In Search of Control" (Douglas McDougall); (12) "Teachers Taking Action: Using the National Mathematics Profile To Improve Teaching and Learning" (Sandra Frid); (13) "Materials To Stimulate Mathematical Thinking at the Elementary Level--A Progress Report on the Kindermath Project" (Ann Kajander); (14) "Tomorrow's Mathematics Classroom: A Vision of Mathematics Education" (Gary Flewelling, Bill Higginson, Geoff Roulet and Peter Taylor); (15) "A Model for the Development of Algebraic Thinking" (Mohamed Mosaad Nouh); (16) "Working towards Curriculum Renewal in Undergraduate Mathematics" (Sandra Frid and Joanne Tims Goodell); (17) "A Conjecture on the History of Mathematical Word Problems: Were the Word Problems Ever Practical?" (Susan Gerofsky); (18) "Desperately Seeking Something: Dilemmas Surrounding the Interpretation of Teachers' Interventions" (Jo Towers); (19) "Scarborough Review of Grade 12 Mathematics" (Lynda Colgan, Peter Harrison and Clara Ho); and (20) "Teaching of Graph Theory for High School and College" (Abraham Bar-Shlomo Turgman). (ASK)

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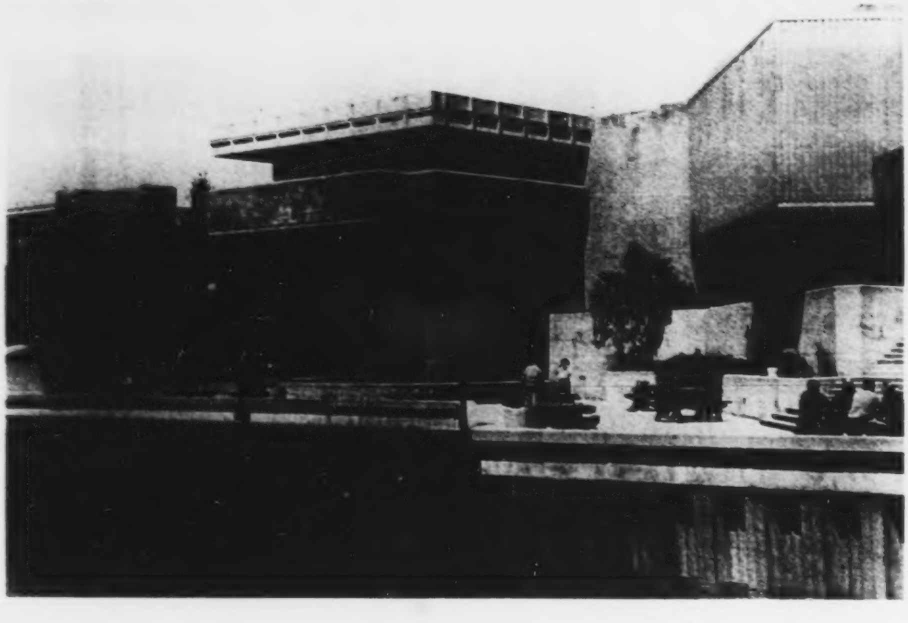
CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS
1997 ANNUAL MEETING



Lakehead University
May 23-27 1997

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CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS
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Lakehead University, May 23-27, 1997

TABLE OF CONTENTS

Editor's Forward *v*
Acknowledgements *vii*
Schedule *ix*
Photographs *xi*

INTRODUCTION *xiii*

PLENARY LECTURES

1. What Does it Really Mean to Teach Mathematics Through Inquiry? 7
Raffaella Borasi, University of Rochester

2. The High School Math Curriculum 11
Peter Taylor, Queen's University

3. Triple Embodiment: Studies of Mathematical Understanding-in-Inter-action in my Work and in the Work of CMESG/GCEDM 23
Thomas E. Kieren, University of Alberta

WORKING GROUPS

A. Awareness and Expression of Generality in Teaching Mathematics 35
Louis Charbonneau, Université du Québec à Montrèal
John Mason, Open University, Milton Keynes

B Communicating Mathematics 53
Douglas Franks, Nipissing University
Susan Pirie, University of British Columbia

C The Crisis in School Mathematics Content 67
Malgorzata Dubiel, Simon Fraser University
David Reid, Memorial University

TOPIC SESSIONS

A Abstract Algebra: A Problems-centred and Historically Focussed Approach 79
Israel Kleiner, York University

B Algebraic Understanding 85
Lesley Lee, Montreal

C Students' Explanations in Linear Algebra 93
Tommy Dreyfus, Center for Technological Education, Holon, Israel

D Mathematics Teaching - How it Could be Done 101
George Kondor, Lakehead University

E Mathematics Teachers' Needs in Dynamic Geometric Computer Environments: In Search of Control 121
Douglas McDougall, Upper Canada College

AD HOC SESSIONS

1. Teachers Taking Action: Using the National Mathematics Profile to Improve Teaching and Learning 135
Sandra Frid, University of New England, Australia

2. Materials to Stimulate Mathematical Thinking at the Elementary Level - A Progress Report on the Kindermath Project 145
Ann Kajander, Lakehead University

3. Tomorrow's Mathematics Classroom: A Vision of Mathematics Education 151
Gary Flewelling, Arthur, Ontario
Bill Higginson, Geoff Roulet and Peter Taylor, Queen's University

4. A Model for the Development of Algebraic Thinking 153
Mohamed Mosaad Nouh, Kuwait University

5. Working Towards Curriculum Renewal in Undergraduate Mathematics 157
Sandra Frid, University of New England, Australia
Joanne Tims Goodell, Curtin University of Technology, Australia

6. A Conjecture on the History of Mathematical Word Problems: Were the Word Problems Ever Practical? 167
Susan Gerofsky, Simon Fraser University

7. Desperately Seeking Something: Dilemmas Surrounding the Interpretation of Teachers' Interventions 177
Jo Towers, University of British Columbia

8. Scarborough Review of Grade 12 Mathematics 183
Lynda Colgan, Peter Harrison and Clara Ho, Scarborough Board of Education

9. Teaching of Graph Theory for High School and College 207
Abraham Bar-Shlomo Turgman, Hebrew University of Jerusalem

APPENDICES

Appendix A Working Groups at Each Annual Meeting 231
Appendix B Plenary Lectures 235
Appendix C Previous Proceedings 237
Appendix D List of Participants 239

EDITOR'S FORWARD

I wish to thank all those who contributed reports for inclusion in these Proceedings. The care taken in preparing a hard copy and disk file of the report, together with camera ready figures made my work as editor a pleasant task. The value of these Proceedings is entirely the credit of the report authors.

These Proceedings will serve to revive the memories of those who participated in the meeting and hopefully will help generate continued discussion on the varied issues raised during the meeting.

Yvonne M. Pothier
Mount Saint Vincent University
August, 1997

ACKNOWLEDGEMENTS

We would like to thank Lakehead University, Thunder Bay, for hosting the meeting and providing excellent facilities. Special thanks are due to Medhat Rahim, Coordinator (Education), Gerry Vervoort (Education), Keith Roy (Mathematics and Statistics), Marilyn Hurrell, Joe Hall, EmiliaVeltri (Lakehead Board of Education), and Bill Otto (Lakehead District Roman Catholic Board of Education) for their time and work prior to and during the meeting to make the experience pleasant and enjoyable for all participants.

Finally, we would like to thank the guest lecturers,working group leaders, topic group and ad hoc presenters, and all participants. You are the ones who made the meeting an intellectually stimulating and worthwhile experience.

SCHEDULE

Friday May 23	Saturday May 24	Sunday May 25	Monday May 26	Tuesday May 27
	0900-1200 Working Groups -Refreshments 1030-1045 1200-1330 Lunch	0900-1200 Working Groups -Refreshments 1030-1045 1200-1330 Lunch	0900-1200 Working Groups -Refreshments 1030-1045 1200-1330 Lunch	0900-1000 Special Plenary - Tom Kieren 1000-1030 Response to plenary 1045-1145 Closing Plenary
1700-1800 Opening Plenary General Introductions 1800-1930 Dinner	1330-1350 Small group discussion - Plenary I 1350-1445 Questions from small groups to speaker 1445-1745 Tour Kakabeka Falls & Terry Fox Monument	1330-1440 Plenary II Peter Taylor 1445-1545 Topic Groups 1 & 2 1545-1600 Refreshments 1600-1730 Annual General Meeting	1330-1350 Small group discussion - Plenary II 1350-1440 Questions from small groups to speaker 1445-1545 Topic Groups 3 and 4 1545-1600 Refreshments 1600-1700 Presentations by new PhD's	
2000-2130 Plenary I Rafaella Borasi 2140 Wine & Cheese Welcome	1800-2000 "Whatshisname" Dinner 2030-2130 Ad Hoc Presentations	1700-2100 Fort William Tour, Dinner and Entertainment	1700-2000 Finlandia Club Dinner & Keterit Dancing Group 2030-2130 Ad Hoc Presentations	

The 1997-98 Executive
Elaine Simmt, Recording Secretary; Lesley Lee, Membership Secretary;
Susan Pirie, Vice-President; Bernard Hodgson, President;
Mary Crowley, Conference Coordinator; Eric Muller, Treasurer

Plenary Session Speakers
Peter Taylor, Raffaella Borasi

Special Plenary Session Speaker
Tom Kieren

Taking advantage of a warm, sunny day
Question period: Peter Taylor's lecture

INTRODUCTION

Je suis très fier, à titre de président du GCEDM/CMESG, d'écrire quelques mots en guise d'introduction à ces Actes de la rencontre annuelle de 1997. Fier, car une fois de plus vous avez entre les mains un document d'une grande qualité matérielle, grâce au labeur de la directrice de publication, Yvonne Pothier—grands mercis, Yvonne! Fier aussi, car ces Actes renferment des textes variés et de grande qualité, dont plusieurs sauront sans doute susciter votre intérêt et favoriser votre réflexion; n'hésitez pas à parcourir les pages qui suivent, vous y trouverez des trésors—et merci à tous les auteurs qui ont contribué au présent document. Fier enfin, car jetant un coup d'oeil à la tablette de ma bibliothèque où reposent les différents Actes des rencontres du GCEDM/CMESG produits au fil des ans, je ne peux m'empêcher de me réjouir à la fois devant l'apport de notre groupe à la réflexion sur l'enseignement et l'apprentissage des mathématiques, et devant le chemin que nous avons parcouru ensemble depuis 1977, année de la rencontre qui, dans la foulée du rapport Coleman sur les sciences mathématiques au Canada, s'est tenue à l'Université Queen's et a mené à la formation du GCEDM/CMESG.

Je ne reprendrai pas ici le récit des origines du GCEDM/CMESG, exposées par David Wheeler dans le document *Les recherches en cours sur l'apprentissage et l'enseignement des mathématiques au Canada*, publié en 1992 à l'occasion d'ICME-7, non plus que la présentation de la "philosophie" sur laquelle repose l'organisation de nos rencontres annuelles, présentation faite par mon prédécesseur, Sandy Dawson, en introduction aux Actes des deux dernières années. Je voudrais cependant mentionner que, bien que nous formions un groupe relativement petit, nous pouvons être fiers du succès de notre activité principale, les rencontres annuelles. La plupart des participants s'entendent pour vanter le cadre assez unique de travail et de collaboration fourni par ces rencontres. Un document tel celui que vous tenez présentement entre vos mains se veut un peu le reflet de ces rencontres—sans prétendre, bien sûr, recréer l'extraordinaire climat d'échanges et d'interaction qui peut y régner.

The annual meetings of CMESG/GCEDM are very successful events, much valued by participants because of their organizational features. They allow, through the *working groups* structure, in-depth treatment of specific themes on which a small group of participants can do "real work" in thinking and reflecting over an extended period of time. They allow, as well, through various semi-formal or informal sessions, the generation of ideas and discussion around developing research projects. They finally provide for each participant an exceptional forum for exchanges of ideas and interactions with others, especially with the two plenary speakers who are, in this respect, *bona fide* participants, working alongside other members throughout the meeting. All these aspects, and others, are crucial to the success of our meetings and to the exceptional working atmosphere encountered there. This can hardly be conveyed through Proceedings such as the ones you are holding in your hands now, but maybe those who know about the group will find therein a little bit of the spirit of our annual gatherings.

En plus du matériel habituel des Actes du GCEDM/CMESG, ce volume renferme un document remarquable sur lequel j'aimerais attirer votre attention: le texte de la conférence donnée par notre collègue Thomas E. Kieren lors de la rencontre de 1997 à l'occasion de son départ à la retraite. Acteur majeur en didactique des mathématiques au plan international depuis de nombreuses années, Tom figure parmi les membres-fondateurs de notre groupe et il en est l'un des anciens présidents (et le principal pilier d'une activité fondamentale de notre groupe, la "tournée des pizzas"). Son texte fournit un témoignage exceptionnel sur l'évolution de certains volets de la recherche en didactique des mathématiques au cours des deux dernières décennies et il met en lumière les liens entre ses propres travaux et les activités du

GCEDM/CMESG. La rencontre de 1997 a d'ailleurs permis aux membres du GCEDM/CMESG, au cours d'un dîner mémorable, de témoigner à Tom leur reconnaissance et leur admiration.

These Proceedings provide a record of the various scientific activities of our 1997 Annual Meeting which took place at Lakehead University, from May 23 to 27. As such, they add to the collective contribution of CMESG/GCEDM to mathematics education and they offer a link with future work of our group.

Bernard R. Hodgson
Président (1997-98)

PLENARY LECTURES

Plenary Lecture 1

WHAT DOES IT REALLY MEAN TO TEACH MATHEMATICS THROUGH INQUIRY?

Raffaella Borasi, University of Rochester, New York

1. INTRODUCTION

To start with, let me briefly address the question "Why is it worthwhile to engage in a discussion about 'What it means to teach mathematics through inquiry'?" I think it is important to clarify that my interest here is not in engaging in a philosophical disquisition about the term "inquiry" so as to determine the "correct" definition for that term, nor to come to some consensus on such definition. Rather, my interest stems from the concern that the consensus surrounding reform documents such as the NCTM Standards (NCTM, 1989, 1991, 1995) may have led us to believe that there is a strong consensus in the field of mathematics education today about not only the need for radical school reform, but also the direction that such reform should take. While I would agree with the first point, I am not sure about the second. Indeed, it is my experience that too many people today use the same terms—such as student-centered, constructivist or inquiry-based instruction—with very different interpretations; so different, in fact, that one begins to wonder if we are trying to achieve the same ends when engaging in school mathematics reform.

I know that I am not alone in this feeling—for example, in a paper she recently prepared for the National Science Foundation, Deborah Ball (1997) identifies the "vagueness" of the vision for school mathematics promoted by the NCTM Standards as one of the main challenges of putting into practice the reform called for by these documents. More specifically, she wrote:

> The air is filled with words about which there has been little discussion—problem solving, understanding, meaningfulness, autonomy, authenticity, *inquiry*. ... Explicating the vision more fully is certainly an important challenge of the reform. And it would help to have more, and better specified, articulations of the ideas and their interpretations. (Ball, 1997, p.80) (my emphasis)

The ultimate goal of my presentation has been to contribute to the kind of articulation called for by Ball. My plan for achieving this goal was to begin by articulating *my own* vision of what should happen in mathematics classes—what I will refer to as a "humanistic inquiry approach to mathematics instruction"—and then invite the participants to do the same in the discussion session that traditionally follows the two main lectures at this conference.

Let me further clarify, however, that my goal has never been to "convince" anyone of the appropriateness of *my* own interpretation of what it means to teach mathematics through inquiry, nor to come to a consensus even just within this group of mathematics educators about what should be the "correct definition" of such an instructional approach. Rather, I hope that by examining in depth some specific examples each of us can all come to make more explicit our own vision for the kind of instruction that we would like to offer to mathematics students (no matter how we end up calling it!) as well as the reasons supporting it. I believe that this, in turn, could contribute more generally to the need for identifying the characteristics of mathematics instruction that we would like to promote in schools.

With this goal in mind, my presentation at the conference consisted of three main parts:

1. a first brief articulation of the theoretical basis for my interpretation of "teaching mathematics through inquiry"—which, given the time constraints, was limited to the identification of the key theoretical assumptions informing my work;

2. an illustration of that approach "in action", achieved by showing a 45 minutes video—almost in real time—of an inquiry experience involving the mathematical topic of tessellation; note that this experience was intended to play the role of a shared example that the group could then examine in depth and refer to as an illustration of, or in contrast to, specific characteristics of mathematics teaching that would be identified and discussed in the course of the lecture and its follow-up discussion;

3. a first commentary on that instructional experience, intended to highlight the characteristics of "teaching mathematics through inquiry" that I thought that example illustrated and, thus, provide the basis for further discussion in the follow-up session.

Given the unusual format of such a presentation, it was difficult to translated it into a written paper for the Conference Proceedings. In the end, I decided to simply report on the main points made in the first and third components of my presentation, along with a brief description of the instructional experience so as to still provide some sort of a shared example (although not as effective as the video itself!). Interested readers could find more information on that experience as well as a copy of the video in the package for teacher educators I have just completed (Borasi & Fonzi, in preparation).

2. A BRIEF ARTICULATION OF THE THEORETICAL FRAMEWORK INFORMING MY "HUMANISTIC INQUITY" APPROACH TO MATHEMATIACS INSTRUCTION

As discussed at more length in other publications (Borasi, 1992; 1996), my own intepretation of what it means to teach mathematics through inquiry has been informed by the following set of assumptions about the nature of knowledge, mathematics, learning and teaching:

- a view of *knowledge* as provisional and continuously refined through a process of inquiry motivated by uncertainty—as suggested by the work of Dewey and Peirce;

- a view of *mathematics* as a *humanistic* and *contextualized* discipline, i.e., as the product of human activity, shaped by personal and cultural values, purposes and context— as suggested by several mathematicians and mathematics educators belonging to the Humanistic Mathematics Network, and by philosophers of mathematics education supporting a social constructivist view of mathematics;

- a view of *learning* as a process of meaning-making requiring personal construction as well as social interaction, and shaped by context and purposes, i.e., a social constructivist view of learning;

- a view of *teaching* as creating a stimulating and supporting environment for the students' own inquiries.

These assumptions are in sharp contrast with those that have characterized traditional mathematics instruction, and which have been often identified as follows:

- a view of *mathematical knowledge* as a body of established facts and techniques, hierarchically organized, context-free and value-free (*logical positivistic view of knowledge*);

- a view of *learning* as the successive accumulation of isolated bits of information and skills, achieved mainly by listening/observing, memorizing and practising (*behaviorist view of learning*);

- a view of *teaching* as the direct transmission of knowledge from experts to novices (*transmission view of teaching*).

In the mathematics education community today there is considerable consensus *against* such a set of the assumptions (often refered to as a "transmission paradigm") as well as about the following alternative assumptions:

- *learning* as a process of meaning-making requiring personal construction *(constructivist view of learning)*;

- *teaching* as orchestrating and facilitating students' constructions.

At the same time, this does not mean that the set of theoretical assumptions I presented at the beginning are necessarily shared by all those calling for school mathematics reform today. Rather, there is still considerable debate at the very least about the following areas:

- the role played by social interaction in the process of learning;

- the influence and roles played by the context in which the learning occurs;

- the interpretation of what constitutes mathematical knowledge;

- the goals articulated for mathematics instruction;

- the nature of the experiences within which students are expected to "construct" their knowledge.

Therefore, it is not surprising that mathematics educators supporting the NCTM Standards and the most recent calls for school mathematics reform may have quite different visions about what such a reform should entail. For this reason, in the following sections I will try to complement the articulation of my theoretical framework with an example as well as with the explicit identification of what *I* think are key characteristics of instruction informed by such a framework as portrayed in my illustration.

3. "EXPLORING TESSELLATIONS TO LEARN "GEOMETRY": AN ILLUSTRATION OF TEACHING MATHEMATICS THROUGH INQUIRY[1]

With the goal of providing an image of what teaching and learning mathematics through inquiry could look like, I would like now to share the key elements of the following "unit" which was designed and field-tested with middle school students in a variety of instructional settings, and was also used to create "experiences as learners" for teachers participating in professional development programs offered at both the in-service and pre-service level. The unit usually takes anywhere between 3 and 6 weeks of instruction in middle school settings, and 4 to 6 hours in a professional development setting.

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[1] NOTE: This experience and the videotape featuring it (Borasi & Fonzi, 1995) were made possible by a grant from the National Science Foundation (no. TPE-9153812 and DUE-9254475). Any opinions, conclusions or recommendations expressed in the video as well as in this paper, however, are solely the author's and do not necessarily reflect the views of the National Science Foundation.

In this unit, the learners explore tessellations to experience the power and excitement of doing genuine mathematical inquiry and to learn some important geometric concepts, such as properties and relationships of specific geometric figures, geometry transformations, theorems and properties about angles, just to mention a few. To stimulate interest in the topic of tessellations and to make it problematic, the learners are given the task of finding examples of tessellations based on this given definition: "A tessellation is the repeated use of any one closed figure that covers a flat surface. The figures do not overlap or have gaps between them." As the learners share and discuss their examples in class, they discover some problems with the definition and, consequently, understand the need to come to some consensus about the meaning and definition of "tessellation" before they can proceed. This preliminary activity not only challenges the learners' conceptions of the nature of mathematical definitions and mathematics more generally, but it also begins to raise questions about tessellations that individuals might want to investigate further.[2]

Once the class agrees upon a definition of tessellation, the learners, in small groups, generate and then explore specific questions and conjectures *of their choice* about tessellations. The instructor supports the groups by modeling how to generate and test mathematical conjectures, making a variety of manipulatives and other resources available, and leading a series of whole-group discussions about the process. Small groups do a final project, such as exploring a set of student-generated conjectures or creating some new tessellation pattern, and present it to the rest of the class. This project offers learners an opportunity to synthesize and demonstrate what they learned.

4. LIST OF CHARACTERISTIC ELEMENTS OF INQUITY-BASED MATHEMATICS INSTRUCTION PROPOSED FOR DISCUSSION

The following list was derived from the in-depth analysis of the Tessellation inquiry experience described above as well as of another inquiry experience on the topic of area. Therefore, I offer this list not as exhaustive of the characteristics of teaching mathematics through inquiry, but rather as a first attempt at identifying such characteristics and as a stimulus for further thinking and discussion.

Also note that, while at the Conference I was able to illustrate this list with specific examples taken from the video the participants had just seen, I do not think that doing so would work here. Therefore, I have instead attempted to articulate *why* I think that these are important elements of inquiry-based instruction in light of the theoretical assumptions stated in Section 2, and then highlight controversial issues within each of these elements that I think require considerable more discussion and study.

1. *Students actively engage in the construction of mathematical knowledge by trying to make personal sense of the mathematical rules, concepts and problems they encounter.* If one accepts the basic tenet of constructivism that learning results from attempts at making sense of situations and requires an act of construction on the part of each learner, indeed this is the kind of behavior that one would wish students to demonstrate in schools. This point does not seem to be controversial among supporters of school mathematics reform.

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[2] The video showed at the Conference featured only this part of the unit, in the case of an implementation that took place within a professional development context.)

2. Students develop ownership of their learning by participating in the generation/choice of the questions and/or problems to be studied. This is an element that is not necessarily shared by all examples of "constructivist-based" teaching, but rather seems more unique to inquiry experiences. Making personal sense of situations often involves generating new problems and sub-questions. The importance of this kind of "problem posing" has been demonstrated by research on mathematical problem solving. Its motivational value is further supported by recent research on motivation that has identified autonomy and choice as important elements for increasing students' engagement in learning and schooling. Yet, it is important to note that "developing ownership" on the content of an inquiry does not necessarily mean that the students need to have full choice on every aspect of the inquiry. Indeed, the role played by *choice* within math inquiry experiences and the different decisions that could be made to this regard by teachers are issues worth discussing further.

3. *Students engage in inquiry not in isolation, but as a community of inquirers that build on each other's ideas and results, and continuously negotiate meanings.* This practice is consistent with the recognition of the important role played by social interactions in learning, and aims at creating a "community of practice" that could both support and "socialize" students to the new expectations of an inquiry math classroom. At the same time, the tension between individual constructions and social interactions within an inquiry experience is an element that may be worth further examination.

4. *Mathematics is portrayed as the product of human activity, i.e., students come to realize that mathematical knowledge (both the one achieved by mathematicians in the past and their own) is tentative and dependent on context and purposes.* As stated earlier, a "humanistic" view of mathematical knowledge is a key element of the theoretical framework I articulated in Section 2. I believe that it is important to communicate such an image to students, as supported by research on the negative impact of a dualistic view of mathematics on students' attitudes and approaches towards this discipline (e.g., Borasi, 1990; Schoenfeld, 1992). Some important implications of this point for me are also that, as an important part of their schooling, students should experience "engaging in mathematics as mathematicians do" and, also, should explicitly examine issues about the nature of mathematics as a discipline—as done for example in our Tessellation unit. Not everybody, however, may agree with such a position and it is indeed worth discussing to what extent and why we would want students to act like mathematicians (e.g., Brown, 1997).

5. *Anomalies, ambiguity and controversy are valued as potential stimuli for inquiry.* This is a logical corollary to Peirce's view of anomalies as the seed for the kind of doubt that can promote questions and inquiry, as well as the constructivist tenet that learning is stimulated by the desire to reduce cognitive disequilibrium. Note, however, that this realization has some radical consequences for instruction: "clarity" has often been held as a key criterion of teacher effectiveness (especially in the case of mathematics teachers); in contrast, I am suggesting that within an inquiry approach difficulties, ambiguity and confusion are not to be avoided, but rather should to be interpreted as the stimulus for inquiry and exploration.

6. *Priority is given to instructional goals such as becoming mathematical problem solvers/inquirers, understanding the nature of mathematics and "big ideas" in mathematics, and developing mathematical confidence.* Arguments in favor of assuming these instructional goals have been presented both on an "economic" ground in reports from influential organizations (like NRC's *Everybody Counts* [National Research Council, 1989]) and a more "cultural" ground by mathematics educators taking a humanistic perspective on mathematics. It is important to realize that this shift in instructional goals is likely to change not only how we teach, but also what we teach and what we value as learning in the mathematics classroom—something that many school administrators and community members seem to find much more difficult to accept than a change in teaching approach and, therefore, need to be more carefully articulated and supported than we have done so far. Also,

such changes in curriculum contents and emphasis needs to be supported by a consistent change in the way we assess and evaluate student learning.

7. *The teacher orchestrates opportunities for students' inquiry and learning by setting up "rich" mathematical situations, and developing activities around them which are meaningful, complex, and open-ended.* Unlike what many people may think at first, inquiry teachers don't just "sit back" and let their students inquire; rather, I would argue that the teacher's job is even more important and challenging within an inquiry approach than in traditional math classes. Note, in fact, that the teacher's responsibility for stimulating and supporting students' inquiries was an integral part of the assumption about teaching characterizing an inquiry approach. The activities that a teacher designs to fulfill this role need first of all to be *meaningful* to the students, so that students can truly try to "make sense" of them as part of their learning experience, and engage with them on a personal level. Meaningful and realistic tasks/situations, in turn, are likely to be *complex*—especially if they are expected to present students with some genuine puzzlement and challenge. At the same time, in order to be accessible to students with diverse backgrounds and abilities, these tasks need to be sufficiently *open-ended* to allow for multiple solutions approaches (and, possibly, multiple solutions as well). How such planning can be effectively done (and learned!) by mathematics teachers is certainly a question that calls for more study.

8. *The teacher facilitates students' inquiries and learning in the classroom through the use of appropriate teaching practices and techniques.* As another crucial dimension of "stimulating and supporting students' inquiry," teachers need to facilitate their students' work as they carry out their inquiry and/or engage in other learning activities in the classroom. To do so most effectively, inquiry teachers should take advantage of the instructional strategies—such as cooperative learning, "writing to learn", etc.—that have been developed and tested in many research studies. Once again, how this can be done effectively it is not straightforward, as sometimes these techniques may need to be modified in order to respond to the different social norms and intellectual demands of an inquiry learning environment.

9. *The teacher listens to students and takes their input into consideration in all pedagogical decisions.* This can be seen as a direct consequence of assuming a constructivist view of learning and a view of teaching as supporting the students' own inquiry and learning. Indeed, if teachers truly value the knowledge learners already possess and try to find ways to build on such knowledge (Barnes, 1985; Confrey, 1991), listening to students seems a necessary first step. It is also an important prerequisite to ensure students' participations in key decisions about a learning activity/ inquiry, which in turn can contribute to the students' feeling of ownership discussed earlier in this list.

10. *Genuine inquiries are long and intensive.* Finally, let me point out how genuine inquiry always take considerable instructional time. This may raise a number of practical issues, not only in terms of "covering the curriculum" (though we know that to be upmost in teachers and administrators' concerns!), but also in terms of gaging students' ability to stay on task and involved over a long period of time.

5. CONCLUDING THOUGHTS

I believe that the principles articulated so far, when taken together, provide a first characterization of what an inquiry approach to math instruction may look like—although I am open to challenges to this regard as well! At the same time, I also believe that there are many alternative and complementary ways to put these principles in practice in the context of instruction.

To conclude, I hope that both the example of inquiry instruction I shared (especially for those who were able to watch the video) and the analysis reported here have provided seeds for further reflection and discussion within this group about how inquiry can play out in mathematics classrooms and how this can contribute to improving mathematics instruction. I also hope that such a discussion will enable each of us to better articulate and refine our vision for the future of mathematics instruction and, thus, provide a contribution to our continuing efforts to make such a vision a reality.

REFERENCES

Ball, D. L. (1997). Developing mathematics reform: What don't we know about teacher learning – but would make good working hypotheses. In S. N. Friel and G. W. Bright (Eds.), *Reflecting on our work: NSF teacher enhancement in K-6 mathematics* (pp. 77 – 111). NY: University Press of America.

Barnes, D. (1985). *From communications to curriculum*. NY: Penguin Books.

Borasi, R. (1990). The invisible hand operating in mathematics instruction: Students' conceptions and expectations. In T.J. Cooney and R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 174-182). 1990 Yearbook of the National Council of Teachers of Mathematics, Reston, VA: National Council of Teachers of Mathematics.

Borasi, R. (1992). *Learning mathematics through inquiry*. Portsmouth, NH: Heinemann.

Borasi, R. (1996)._*Reconceiving mathematics instruction: A focus on errors*. Norwood, NJ: Ablex.

Borasi, R. And Fonzi, J. (in preparation). *Introducing teachers to an inquiry approach to mathematics instruction: A multi-media package for teacher educators*.

Borasi, R. and Fonzi, J. (1995). Teaching mathematics through inquiry: What does it look like? (videotape).

Brown, S.I. (1997). Thinking like a mathematician: A problematic perspective. *For the Learning of Mathematics (32)*, 36-38.

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In E.von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 111-138). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Connell, J. P. and Wellborn , J. G. (in press). Competence, autonomy, and relatedness: A motivational analysis of self-esteem processes. In M. Gunnar and L. A. Sroufe (Eds.), *Minnesota symposium on child development*, 22. Hillsdale, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (1995). *Assessment standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Research Council (NRC). (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, D.C.: National Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Plenary Lecture I

THE HIGH SCHOOL MATHEMATICS CURRICULUM

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No invitation to speak is so wonderful as that which comes from your own group. I feel privileged and honoured to be here tonight among so many friends and colleagues who represent so many years of fine work that we have done together—in fact twenty years of fine work if I have my numbers right. I think 1977 was the first CMESG meeting I attended. It was I believe the founding meeting of the organization.

On the down side, there's a problem talking to a body like this about math education. Everything that I might say, you already know. On the other hand, perhaps it is simply my responsibility, in standing here tonight, to find the words that we would all speak, and to say the things that we all feel should be said at this time in the life of our country and our world.

I have just recently put out this high school text that you may have been admiring or criticizing, hopefully some of each, and although it is only the first of 101 drafts, it does in fact say much of what I would try to say. There's much more to be done—conceptual gaps to be filled, technical results and exercises to be assembled, etc. I must face the question of how this document is to fit into a whole curriculum.

The form of this book has been strongly influenced by an analogy—one that, as most of you know, I have been talking about for a few years—an analogy between the shape of curricula in the sciences and the arts, more specifically between mathematics and English. There are two main factors here, one has to do with *organization* and the other with *sophistication.*

Curricula in the arts and humanities are organized around works of art, and that is the organizational principle behind this book, whereas in the sciences, especially in mathematics, they are organized around technical skills. If this book were to be used in today's classroom, teachers would require a detailed "map" of its technical terrain. One can imagine requiring such a map in the study of literature, but teachers generally do not feel the need for this. This actually gives them a certain amount of professional freedom that math teachers generally don't have. With such freedom comes a certain responsibility, of course, but these open the way to real professional growth.

Curricula in the arts and humanities work at a much higher level of sophistication than is usually the case in the sciences. And most of the problems in this book are more sophisticated than we would normally find in a high school math classroom. Some teachers have told me that the problems are too "difficult," but "sophisticated" is a better word. Richard III is sophisticated—but there are easy things that a student can do with this play, and there are difficult things. It depends on where you are and what you want. The same is true of these problems.

For many years I have struggled with the question of style—how are mathematics problems to be written?, how are we to model inquiry, exploration?, and for whom, teachers or students, or both? This is still an ongoing question for me, and the various styles I have played with in this book are still tentative and experimental.

Two weeks ago I presented this book at the annual meeting of the Ontario Association of Mathematics Educators (OAME), the professional association of high school math teachers in Ontario. In two different sessions I worked with two problems, *Sum of cubes* and *Trains*. I got many interesting reactions. One teacher compared me to Da Vinci. "Your book can be compared with Leonardo's sketches of airplanes. The good part is that they gave us the idea that we could fly. The bad part is that those particular planes could not fly." He went on to invite me into his classroom, or better yet to spend the next four years at his school coping with a full teaching load instead of the paltry two courses I have at the moment which are furthermore delivered to students who are already committed to their studies. Fair comment.

Indeed, these are interesting times in Ontario in high school curriculum reform. There's lots of idealism in the air, lots of frustrated teachers, and a government who is determined to make some real changes, though the shape of the final outcome of all this is quite unclear. One thing that is receiving a lot of attention all of a sudden is the idea of a problems-based curriculum, and that's nice for us because we have been advocating a movement in this direction for many years, even before the NCTM Standards emerged.

The various debates that I witnessed or presided over at OAME seemed to focus on five questions.

1) How does the organization and mastery of technical skills fit into a problems-based curriculum?
2) Are these problems accessible to all students?
3) Are these problems accessible to all teachers?
4) How do we evaluate the students?
5) How do we evaluate the curriculum?

What I will do is comment on all these questions, but not in a comprehensive way and not quite in the above order. The issues at stake here are ones that we have all talked about again and again, and we have come to know one another's views so well, that I'm sure you could all go away and write out exactly how I would try to respond.

But first let me run very quickly through an example from the book, just so we have something to hang the discussion on. Most of our fellow citizens do not entirely understand what mathematics is, for example, they sometimes confuse it with arithmetic. Now mathematics and arithmetic have a close and wonderful relationship, and to set the stage for my subsequent remarks, I will take a moment to examine this relationship, using the first problem in the book—*Sum of cubes*. This is appropriate as the problem was given to us exactly a year ago in Halifax by Ed Barbeau in his topic group.

Start by studying the following tables (see figure 1). We are struck that the sum of the second column is the square of the sum of the first. This nice pattern emerges from a collection of summation formulae that we all met long ago (see figure 2). The sum of consecutive integers, their squares, and (perhaps) their cubes.

The last of these is less well known but it has the striking property that the sum on the right is the square of the first sum, and that certainly fascinated me when I first laid eyes on it. And of course I immediately wrote it as:

$$\boxed{1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2}$$

the sum of the cubes
is the square of the sum

numbers	*cubes*
1	1
2	8
3	27
6	36

numbers	*cubes*
1	1
2	8
3	27
4	64
10	100

numbers	*cubes*
1	1
2	8
3	27
4	64
5	125
15	225

Figure 1

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

$$1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Figure 2

And then at our meeting last Spring in Halifax, Ed Barbeau presented a remarkable divisor construction. Ed asked for a number from the class (with lots of divisors) and got 72 (see figure 3). He then listed the divisors of 72 (col. 1) and then the number of divisors of each divisor (col. 2). Then he remarked that the second list has the property that "the sum of cubes is the square of the sum."

Table for $N = 72$

Divisors of 72	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
8	4	64
9	3	27
12	6	216
18	6	216
24	8	512
36	9	729
72	12	1728
SUM:	60	3600

Figure 3

There's that curious property again, and right away a mathematician wants to know how it relates to the more familiar example we already have. Is this really a new result, or is this just the old property dressed up in unfamiliar clothing?

The first thing to do is to look at a number of simple examples (see figure 4).

Well, there are certainly some patterns to be accounted for. For example, the N=5 table is small, having just a 1 and a 2. When will that happen?—*precisely when N is prime*. There's one result. But the striking observation belongs to the tables for $N = 8$, 9 and 16. For these, column 2 is a set of the original type (the integers from 1 to n) and so the "sum of cubes" property we have here is just the original one. Well now that's *very* encouraging. But there's more—Look! *Look at tables 8 and 9. What are the sums?—10 and 6. And their product is 60, which is the sum for 72. And the column 3 sums of 100 and 36 also multiply to give 3600 which is the column 3 sum for N=72. And 8 and 9 of course have product 72. Holy cow.*

Noticing these patterns, appreciating their possible significance, and tracking them down–that's what a mathematician does; that's when arithmetic becomes mathematics.

First of all, when will column two have the familiar form—the integers from 1 to n? The answer is that this will happen when N is a prime power. If $N = p^k$, for some prime p, then column one will consist of the powers of p from 0 to k (these are the divisors of p^k), and column two will therefore consist of the integers from 1 to k+1. But when N is not a prime power (e.g., 72), column two is of a different kind and we want to see how this case fits in with the familiar one. And we start with the fact that 72 is a product of prime powers 8 and 9. Does this make the 72-table in any sense the *product of* the 8- and 9-tables?

Table for $N = 5$

Divisors of 5	# of divisors	cubes of col 2
1	1	1
5	2	8
SUM:	3	9

Table for $N = 9$

Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

Table for $N = 8$

Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

Table for $N = 10$

Divisors of 10	# of divisors	cubes of col 2
1	1	1
2	2	8
5	2	8
10	4	64
SUM:	9	81

Table for $N = 12$

Divisors of 12	# of divisors	cubes of col 2
1	1	1
2	2	8
3	2	8
4	3	27
6	4	64
12	6	216
SUM:	18	324

Table for $N = 16$

Divisors of 16	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
16	5	125
SUM:	15	225

Figure 4

We need to start by asking how the divisors of 72 are related to the divisors of 8 and 9. And because this is a prime power decomposition, the divisors of 72 are exactly the products of the divisors of 8 and the divisors of 9. It's important to be careful here. Certainly if I take a divisor of 8 and a divisor of 9 and multiply them together, I'll get a divisor of 72, but I'm saying more than that. I'm saying that if we make a list of the divisors of 8 and of the divisors of 9, and then take all possible products of one list with the other, we'll get exactly the divisors of 72, with no repeats. [To make this argument, we need some lore here about how the divisors of a number are obtained from its prime factorization—a good chance for some careful technical work.]

So now, let's ask about column two. To take an example, consider the divisor 12 of 72. I ask how many divisors 12 has, but I am going to try to find the answer, not by looking in column two in table 72 (where there's a 6 sitting right beside it) but by trekking over to tables 8 and 9. To do that, I write 12 as a product of prime powers: 12 = (4)(3) so we look at the 4-row in the 8-table and the 3-row in the 9-table: Now column two in those two tables tells us how many divisors there are of each factor—4 has 3 divisors and 3 has 2 divisors. How many divisors does that give for 12?—well, (3)(2)=6 because the divisors of (4)(3) are all the products of the divisors of 4 and the divisors of 3.

What has this told us?—*that each column-two entry in the 72-table is the product of the corresponding column-two entries in the 8- and 9-tables.*

To emphasize this, I rewrite the 72-table replacing the entries by the appropriate products. This also prompts me to reorder the rows to make the structure easier to see.

Table for $N = 8$

Divisors of 8	# of divisors	cubes of col 2
1	1	1
2	2	8
4	3	27
8	4	64
SUM:	10	100

Table for $N = 9$

Divisors of 9	# of divisors	cubes of col 2
1	1	1
3	2	8
9	3	27
SUM:	6	36

Table for $n = 72$

Divisors of 72	# of divisors	cubes of col 2
1=1·1	1=1·1	1=1·1
2=2·1	2=2·1	8=8·1
4=4·1	3=3·1	27=27·1
8=8·1	4=4·1	64=64·1
3=1·3	2=1·2	8=1·8
6=2·3	4=2·2	64=8·8
12=4·3	6=3·2	216=27·8
24=8·3	8=4·2	512=64·8
9=1·9	3=1·3	27=1·3
18=2·9	6=2·3	216=8·27
36=4·9	9=3·3	729=27·27
72=8·9	12=4·3	1728=64·27
SUM:	60=10·6	3600=100·36

Figure 5

The way we have written the 72-table shows in a very precise sense what it means to assert that this table is the "product" of the 8-table and the 9-table. What this shows is that the "sum of cubes = square of sum" result for column 2 of the 72-table, follows directly from the familiar result for the sum of consecutive integers—as found in column 2 of the 8- and 9- tables.

The calculations can also be set out in equation format, and in fact this form is probably the orthodox mathematical approach, but the table argument is quite rigorous and more transparent to me.

What's the bottom line?—there's lots of arithmetic here and there's lots of mathematics. Each one by itself is arid and without meaning. But together, they explode into life.

1) *How does the organization and mastery of technical skills fit into a problems-based curriculum?*

I'm not going to answer this question. The reason is that it's not problematical—it's something that has to be done, but we can all imagine how to do it. The real problem for me is that the person who asks me this question has decided that I don't think technical skills are very important and that makes him uncomfortable for all sorts of reasons. Isn't it just what he'd expect from some high and mighty university person who sets himself above the bread and butter, well at least the bread, of his modest high school classroom?

So I will say this to him. What I have done is to put problems rather than skills at the centre of the curriculum. Now you think that I have done this because I do not think very highly of technical skills. But it is the reverse that is true. I have done this because I value these skills even more highly than do you. Without technique I could not move forward in my work. When a new technical result comes along, and this does not happen quite as often as you might think, it is celebrated by the whole community because it opens up new possibilities. Technical skills are so wonderful that they deserve a context to allow students to see this wonder for themselves. And that's what we have done here. So though it may not appear so to you, this book actually celebrates and honours technical skill.

3) *Are these problems accessible to all teachers?*

A teacher must serve two masters—her subject and her students. So that the teacher who says with pride: "I don't teach math, I teach students!" has only described half her job. It's worth noting why she might have proclaimed this—she's trying to demonstrate the human face of her classroom, and she has failed to find that in her subject matter. If she could have found that, she would have encountered no essential conflict or tension between the two masters.

I'll say something here about serving mathematics, and I'll talk about serving the student under question 2.

Elizabeth, who was in my working group at OAME asserted that 8 out of the 10 mathematics teachers at her high school would never use these problems in their classroom and would object strenuously if they were required to do so. You have no idea, she said, looking at me sympathetically.

Certainly we must change our expectations of what teachers need to do with these problems. The analogy with English, in which the teacher almost always works with material that is beyond her in its level of sophistication, is only partial, but there are some valuable insights we can gain from it. It is not true that the student needs to master these problems, nor is it true for the teacher. It is enough that the teacher be aware of the importance and the sophistication of the material, be able to appreciate it, work with it, think clearly and honestly about it, and understand the ideas of her students. [Having said that, this is still quite a bit to ask of the teacher.]

And what can we do to help them in this? There has been lots of talk of new forms of professional development written right into their contract, gigantic ever expanding world-wide webs of rich resources, computers in every classroom. But actually the first thing they need from us is to get off their case. Already they serve too many masters—government, university professors, principals, parents, guidelines, text books… We all know what it's like to serve a multitude of conflicting or even if not conflicting simply noisy demands or even if not demands simply suggestions made in the quietest most respectful way. If we have our professional integrity intact, as most of us do, then these suggestions and even these demands are not threatening and need not limit our freedom. But otherwise they can be real burdens, and can erode what little integrity we have managed to garner over many years of teaching.

Once a teacher has managed to get some control over her own working environment, she will need some help from us, for example to discover how to get some richness out of these problems in an elementary way—without feeling that she has to completely solve them in the conventional mathematics sense. As Judith Fonzi remarked in Rafaella's tape: *we can't teach it differently if we haven't experienced it differently.*

And what of Elizabeth's 8 teachers? She's probably right—I have no idea. I think that many of you here have a better idea than I.

2) *Are these problems accessible to all students?*

Teachers assure me that these problems are much too hard for most of their grade 12 students let alone grade 10. And for their general students—well, forget it.

As you have seen, these are mathematics problems. Perhaps they are telling me that mathematics is too hard for their students. Maybe so. I'm not sure. But if that's the case, then we should not advertise ourselves as math teachers but teachers of something else. And maybe we should be happy to teach something else. Maybe our students don't need math, maybe most of them need something else—something closer to plain arithmetic.

Of course I feel that our students do need mathematics, not so much to get by in their job or the day-to-day business of life, but for other reasons—reasons having to do with what it means to be fully human—and we teachers of mathematics have a responsibility to provide it for them.

Just last week I talked for a while with one of our M.Ed. students, Lorayne—an experienced teacher who is currently tested to the limit by her alternative math class at Gananoque Secondary School. These students represent a huge array of abilities and backgrounds and share only the distinction of having been unable to make it in a normal classroom. We agreed that in many ways those students need these problems perhaps more than anyone.

Mathematics has the power to elevate. It gives us a particular glass through which to see the world, and to see inside of ourselves. There are of course lots of glasses with different hues and points of view. But mathematics constitutes a pole, an endpoint of a fundamental continuum. In its purity and its independence, it has an easy friendship with a world it really does not need. Of all disciplines, it is the outcast. Of all the subjects it is the one that can thumb its nose at the world if it wants—hey, I don't need anything from you, I can get along just fine, so leave me alone will you? just leave me alone! For this reason it is needed most of all by the outcasts of our student body. It has the capacity to provide light and hope to those who are mired in social or cultural confusion. So in many ways the students who need mathematics most are the ones from whom it is withheld. Not responsive enough to cope with an advanced stream, they are relegated to job-oriented skills that machines can perform. They must feel unspeakably insulted. And beware of deciding that they are not clever enough to engage these problems.

This is a difficult point and I want to say more about it. James, my son of 18 years, is struggling for the third time with grade 12 general English—this time he really needs to pass it. He's also taking OAC physics—just why I am not sure as he has no other OAC's and is unlikely ever to take another. His heart is not in either of these subjects. In fact his heart has not been in anything in high school except the one term in grade 12 in an alternative program where they spent half their time traveling, talking with the native people who live beside the James Bay Hydro electric dam, living in the dead of winter with Inuit families on Baffin Island, and trading guitar melodies with the miners in Nova Scotia.

James, don't you think we might do a couple of physics problems tonight? It would really be a great idea to pass this course.

Silence. *Yeah, I guess.*
Well, I'm around.
But a short while later he's on his way out.
Got a band practice over at Casey's.
Okay, see you later. Don't be too late, eh?
I won't. Hey, Dad?
Yeah?
Alright if the band practices here tomorrow night? In the basement?
Sure. You know the rules.
See ya.

He knows the score as well as anyone. He sees me in my office. He sees me always (obsessively) typing away at home. He sees computers everywhere. He knows that the world belongs to those who embrace science and technology. He knows all of that.

But what world? Exactly what world? He talks about James Bay as if it was named after him. Funny 'cause he was named James Taylor because his namesake was singing *Sweet Baby James* when he was born. Lots of worlds there. When I talk to him about the world, exactly which one do I have in mind? Our view of the world is so tied up with bigger faster better, that we hardly recognize the critical assumptions that lie behind our educational pronouncement. That *lie* behind our educational pronouncements.

That whole technology thing—the LIST of technical skills. My god, you need math to do everything—your income tax, figure out your mortgage, and if you don't know your way around computers, you'll be lost. Listen to us! Just listen.

It may all be true, but *he* knows all that. And anyway the worst thing is probably for me to tell him what he already knows.

What can I do for him? Well for starters I can get off *his* case too. I've done that already— being after him all the time just doesn't work. My duty to him as a parent is to show him my true self, and try to behave in a way that accords with this. For me that means that I have to reveal to him the artist that is inside me, the best and the worst and hopefully more of the former. And in turn I have to try to understand the one that is inside him. And we do this with an exchange of art—my mathematics, his music, my oatmeal raisin cookies with cranberries instead of raisins, his toasted egg and bacon and cheese and mayonnaise and salsa sandwiches. And for the rest, I have to trust him to figure out the future for himself.

For me, one striking aspect of James has to do with focus—when he has it and when he doesn't. At the moment in high school we spend a huge amount of time trying to teach ideas and skills to students that they are not prepared to focus on. And for this task focus is almost everything. So it's a huge waste of time. For us and them. Whereas if we bring art and music into the classroom, the students who are focused will actually perceive and master the skills. Maybe the ones who aren't will find a reason to change.

The last two questions are difficult. They are tied up with ideas of standards and certification, and I don't really have answers to most of that.

4) *How do we evaluate the students?*

In a course last year I gave the *Handshake* problem as the second assignment. The problem is that you have *n* people in the room and they all shake hands with everyone else and you ask for the minimum time required to get all the shaking done if you can only shake with one person at a time.

Well one student, Joanne, who is a visual artist, submitted a video tape of a native community having their handshaking ceremony where you go around in two circles, and shake everybody's hand. And getting it done in the minimum time really isn't the objective here at all. By some quirk of fate this happened to be the same native community in Ouje-bougoumou that my son James had visited the year before. Joanne then wrote an essay on the significance of the ceremony. At the end of it she included an apology. I'm sorry—I know this isn't what you wanted. I just don't have the technical skills to answer your question. But I do enjoy the classes a lot.

How am I to mark that? That's not actually an easy question. There was a little arithmetic in her answer, a smidgeon of mathematics, but mostly it was other stuff. Technically she was not ready for the course, but the course was a wonderful experience for her, much better than a more basic course with a narrow technical aim.

It is certainly true that as we broaden the scope of our math courses, we will have to broaden the criteria of evaluation.

Generally I don't worry so much anymore about the certification aspects of the marks I assign. The teacher on the next rung up will want to demand that a 75 mean a 75 as far as readiness for his subject is concerned, but in my experience he is as often likely to abuse that information as to make proper use of it, and so that's one thing I am happy to let go of just a bit. And I would give senior high school teachers the same advice.

What's important is that the student is not fooled—and many *are* being fooled right now. The mark I gave Joanne was not great but it was a pass. Even so she was not fooled by it. And if she goes out into the world thinking that mathematics is a bit different from what it really is, well, she's got lots of company. Actually she just might have a better idea than most.

5) *How are we to evaluate the curriculum?*

Let me give two examples of the debate around this question, both of which I encountered at the OAME conference, and both concern the NCTM "Standards." The first is from Stephen Leinwand who is the president of NCSM (the National Council of Supervisors of Mathematics) and this is the lead article in their July 1996 newsletter. By the way, John Saxon is an author and advocate who writes sometime successful "how-to" math textbooks.

> I was as disgruntled as anyone with the barrage of obnoxious ads that John Saxon ran during the past year, but those of us committed to reform have to admit that we have left ourselves wide open to John's attacks. Like him or hate him, John Saxon is not wrong when he asks us 'Where is your data?' and 'Where is the evidence of your success?' He's not wrong when he taunts 'if gains were possible, the NCTM would have been able to find a teensy-weensy gain somewhere and would have shouted the results to the world.'
>
> So I think we ought to thank our Number One Critic for reminding us that it's time to prove our case with data and with real results. I believe it's time for us to begin providing clear answers to his questions and to begin showing the critics, the naysayers, and the skeptics that this reform movement is, in fact, beginning to make a difference in student achievement.

> Let me start the ball rolling. One of the new NSF-funded secondary mathematics curriculum development projects is the Math Connections project being developed under the leadership of the Connecticut Business and Industry Association's Education Foundation and the Hartford Alliance for Mathematics and Science Education. Like its sister projects, it is an attempt to create classroom tested materials that reflect the spirit of the NCTM Standards. That means a significant focus on realistic problems and narrative answers, a reliance on graphing calculators, and integration of content, and a significant reduction in symbol manipulation skill— all the components that John Saxon abhors! So, after two years, is such a program working?

Well, what they did was to take two groups of students in grade 8, tested them and found no significant difference between the groups, gave one group the traditional math program for two years, and the other group the reform program, and at the end of grade 10, the reform group tested significantly better. I assume the test was reasonable, etc. So Leinwand continues:

> Yes, this is but one small picture of reform. But it is also one clear picture of success that needs to be expanded and replicated and then joined with similar success stories that arise across the country.
>
> So I challenge each and every one of you to begin gathering the data and sharing the results of reform at work. This Newsletter is an ideal place for this sharing to begin. What better way to channel your anger at John Saxon's cheap shots than helping to build the case about how wrong he is and how well our "Standards" are beginning to work!

The next article was given to me by a teacher who I have known for some years. As he was leaving my talk he pressed it into my hand as said "you oughta read this." It's the transcript of an invited address to the California Board of Education last month by E.D. Hirsch, Jr. After three pages we get this:

> Let me turn to math education. I read a recent report in Education Week which stated that there were two rival math groups in California vying for your approval. On the one side there is what Education Week called the "reform" group who want to put in place the standards of the National Council of Teachers of Mathematics (NCTM), and on the other, the so-called "anti-reform" group that calls those standards variously "fuzzy math" and "whole math." I thought that the tone of the Ed Week report was typical of current educational reporting in that the NCTM approach, which reflects the dominant view among educators, was labeled "reform" while the dissident group that is trying to effect change was labeled "anti-reform." That kind of ideological bias in reporting is characteristic of the education world, and it well illustrates the need for constant vigilance.
>
> To this Board I hardly need to restate the details of the math debate. The NCTM group stresses conceptual understanding over mindless drill and practice, while the dissident group stresses the need for drill and practice leading to mastery. To resolve the issue, which researchers should you listen to? Here are three suggestions: John Anderson, David Geary, and Robert Siegler--three highly distinguished scientists in the psychology of math education. What are they likely to tell you? I believe you will get strong agreement from them on the following points: that varied and repeated practice leading to rapid recall and automaticity is necessary to higher-order problem-solving skills in both mathematics and the sciences.... They would provide you with reliable facts, figures, and documentation to support their position, and these data would come not just from isolated lab experiments, but also from large-scale classroom results. If these two scientists agreed on all these

points, that is the consensus you should trust, no matter how many pronouncements to the contrary might be made by national educational bodies.

In both articles science is invoked to make a case. Leinwand cites a statistical study which shows that students in a reform program test better than students in a traditional program. And Hirsch mentions the word "scientist" again and again and urges the board to heed the advice of the "top scientists" and even quite generously tells them who these might be.

Well I am a scientist myself. I am awed by the power of science to unlock the mysteries of the world, not only physics, but behaviour, even that of us humans. Science has been successful, so successful that there is a certain *cachet* in being called a scientist. With such success, we would certainly want the tools of science to be employed in the evaluation of our curriculum.

But let's be careful. The point for me is this. What if Leinwand's tests had come out the other way? What if the reform students had done worse? It wouldn't change my view that these are the problems I should be giving my students. I'd likely ask exactly what I think I'm doing and how I'm doing it, what really are my objectives, exactly what those "tests" are actually measuring, etc. and science is an excellent tool for that process—forcing us to ask good questions, keeping our concepts precise, and keeping us honest about our answers. And it might well lead me to better ways of writing of presenting of organizing, to a better pedagogy. But if you ask me if there is there any test I could devise whose negative result would convince me that the material is wrong I would answer no. It is an unfalsifiable hypothesis. And therefore it's got nothing to do with science. Ah, you will say, then it is simply a religious belief. I think it is not religion but art. When an artist is sure of his work, a viewer-response test does nothing to convince him that the work is wrong, though it may be an occasion for him to question his objectives or his relationship with others. The knowledge, of what is right, comes from a deeper place than science and we must honour that. Science has a lot to offer us in this quest for a new curriculum, but it cannot change what I know mathematics to be.

In an e-mail message Lorayne sent me right after our talk she said:

> Beyond that, one of the closing remarks you made has relieved a even more pervasive and onerous burden. You assure me that I don't need to worry my head about justifying my approach. When I share with students the fascination I have in the relationships and the mathematical wonders imbedded in all aspects of our world, I 'know' we are doing real mathematics. The dilemma to date has been how to justify this approach. I can provide empirical evidence to demonstrate that attitudes towards mathematics, science, art, history, and poetry, for example, change dramatically in a relatively short time when we 'play' with relationships. I have neither been able to afford the time to justify this position nor had any real clues as to where to start.

One important thing that Lorayne has going for her is that she really does have the freedom to do what she feels is right—because society has already in some sense written her students off. Thus the door is open for her to develop as a true professional. Happily for her students, she understands better than most of us that we mathematicians have an important role to play in helping them to become fully human. Not only for the richness of their own lives, but for the quality of the conversations that go on in politics, business, law, medicine, and even education.

Specila Plenary Lecture

IMAGINING EMBODIED MATHEMATICAL KNOWING: SOME THOUGHTS ON MATHEMATICAL KNOWING IN ACTION, INTERACTION AND IN THE CMESG/GCEDM

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ABSTRACT

Mathematical knowing is very often associated with the noun "mathematics" - that is, as something captured in a text or in expressed form on paper. To the extent that it is thought of as a process it is frequently thought of in personal terms as a construction or an acquisition or perhaps as problem solving. In this essay I look at such knowing not as "in the book" or "in the head" but in ongoing embodied actions and inter-actions in the temporal "now". To think about such knowing or such embodied actions is to interrogate them for their intra-personal structural dynamics; inter-personal social and environmental inter-actional dynamics; as well as the mathematical/cultural dynamics of the practices occurring in such actions/inter-actions—*all-at-once*. I have been characterising this stance as thinking about mathematical knowing in action and inter-action as *triply embodied*. Such an interrogative and interpretive stance is related to a particular view of mathematical knowing. Although such knowing includes performing discrete mathematical acts such as constructions, computations and proofs and indeed involves engaging in solving particular problems it is more than that. Using ideas derived from Maturana and Varela[1] mathematical knowing on this view is the bringing forth of and dwelling in a world of mathematical significance with others in a sphere of possibilities.

Because it is beyond the practical scope of this brief essay to allow for the perusal of mathematical actions I am taking the liberty of asking you to imagine several examples of such actions and then to consider my triply-embodied interpretations in light of your own imaginings. These examples are based in actual events. But what you read here are only my brief imaginings and not a reportage of such events. These imaginings include with my imaginings and interpretations of the on-going and historical functioning of our organization from this triply embodied perspective.

This essay concludes with a postscript in which I briefly trace the genesis of the sense in which I imagine these mathematical knowings and the genesis more particularly of the triply embodied view of mathematical knowing that I bring to my interpretations.[2] What follows is both an essay and a

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[1] H. Maturana and F. Varela, The Tree of Knowledge. Boston: New Science Li brary. 1987.

[2] I would like to thank CMESG/GCEDM and especially the 1997 executive for giving me the opportunity to make the presentation at the Thunder Bay meeting. I could think of no greater honor that I have ever received or could receive. In the spirit of that presentation - this essay is not a research paper. Although like much of my professional life in action the writing of this paper represents research for me, this paper has none of the trappings of a typical research paper. I offer it simply as possibly providing

commentary—it is aimed to inform you about a particular view of mathematical knowing. But it is also a personal commentary on that view, on our organisation, and on my own intellectual history.

IMAGININGS 1: FOOTPRINTS OF MATHEMATICAL KNOWING

Ralph Mason had contrived an activity—which he brought to Thunder Bay—so that his pre-service teachers could both enter a space of mathematical possibility themselves and could also observe and interact with children while those children acted in such a space. The activity goes something like this: Imagine that a child has 27 2 cm cubes; imagine further that the child builds a "garage" that is a cube 8 cm on an edge with an open door on one face. The general task facing this child—perhaps in inter-action with others and possibly a teacher—is to build objects by gluing together the 27 cubes in such a way that cube faces match and the created objects fit in the "garage". The child is then challenged to re-present and describe such objects, to compare each with other such objects, and to make up mathematical statements about them.

As might be expected with either 10 year old children or pre-service teachers, it might be useful to prompt mathematical activity with suggestions or questions. Some of Ralph's questions resembled the following:

- Can you draw a 3-d sketch of your object using dotty paper or grids?
- Can you draw front, side, and top views?
- Can you describe the "layers" of your object?
- If you gave some or all of this information to a friend could they build your object without seeing it? What information would you think to be most useful to them?

Now imagine that I was talking to Ralph and Lars Janssen about the activity and engaging with them in some mathematical play with "garageable" forms. After some minutes I left and ran into Elaine Simmt. Because of her interest in parents and children working together on mathematical tasks I suggested that she see Ralph's activity. She went to Ralph's room and was joined by John Mason in building and studying objects. They became especially interested in the "footprints" of the objects (the mark the faces of the object would make on a sheet of paper if it were dipped in a paint or ink of some kind). David Reid inserted himself into what was now a small roomful of mathematics in action. He convinced the others to think about objects with of up to six fold "footprint symmetry"...

Now let your imagination move to one of my favourite research venues, our off-campus pizza-run location. Elaine has now drifted off to talk about the role of history of mathematics in her parent/child research. But David and John, cubes in hand are hard at it in "footprint" space. Tommy and Marty join in and Marty questions the restriction of glueability. That is

- must cube faces in an object be glueable face-to-face?
- will glueable edges or vertices do?
- what impact does this have on footprints and symmetries?

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occasions for your further thinkings about mathematical knowing.

After more than an hour of long range kibitzing Frederic adds some topological insights and queries, but then abandons the group which has grown in number and has spread cubes and objects across the table and their imaginations. Many theorems later the pizza is gone but the mathematics lingers...

This brief imagining prompts me to ask some questions which have formed an important part of the agenda especially of some of the working groups and topic groups of CMESG/GCEDM over the years:

- Where was the mathematical cognition in all of this and how might such cognition(s) or knowing(s) be characterised?
- Where was and what was the mathematical curriculum[3] in all of this?

So where was the mathematical knowing?

- Was it simply in the heads of each of the participants in the scenes that prompted my imagining?
- Was it in the blocks and questions posed on paper? That is, was it in "the book"? Was it predetermined by the materials and questions? Neither of those two points of view seems to satisfactorily capture the knowing as I remember it and have imagined it. Nor does either view capture mathematical knowing in general. Following Maturana and Varela, it seems more useful to think of such knowing as the *bringing forth of a world of mathematical significance with others and the dwelling in that world.* Rather than look at that which was mathematical in terms of abstract and disembodied "ideas"—for example, the idea of "glueability"— I am suggesting that we think of the mathematical in terms of fully embodied personal mathematical knowing. I am further suggesting that such knowing is embodied in three ways all-at-once.

The first embodiment focuses on the person in action and asks how that person's structure or lived history determines the person's mathematical actions and how that structure is changed by the actions. That is, the first embodiment is concerned with *personal structural dynamics*. How was the person thinking/acting and how does thinking/acting change?[4]

The second embodiment recognizes that the person is acting in an environment and in interaction with others. Mathematical knowing does not simply emerge as a product of brain or other personal functioning. It is co-emergent. The environment and others in it are fully complicit in such knowing. This embodiment calls our attention to the nature and role of the *inter-actional dynamics* of mathematical knowing.

The third embodiment, embodiment in the body of mathematics is a take off from the work of Brent Davis who first brought it to my attention in those terms.[5] A person acting mathematically is embodied in the mathematical practices of her or his surround such as the classroom or provincial or mathematics departmental syllabus but such actions can also be viewed in relationship to the historic and contemporary practices of mathematics. The embodiment in the body of mathematics prompts us to think about the *cultural dynamics* of mathematical knowing in action. As suggested above since I am thinking of mathematical knowing in terms of building and living in a mathematical world then it is both useful and necessary to think of the personal/structural; the inter-personal/environmental inter-actional; and the mathematical/cultural dynamics all-at-once however hard that is.

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[3] Susan Pirie in our working group was careful to distinguish the curriculum from the syllabus. I am adhering to this distinction here and not trying to identify a list of topics or objectives or ... underlying the imagined actions. Curriculum as used here points to the "curare", the running - the knowing in action and the setting for that action.

[4] Susan Pirie and I have been thinking about and researching along these lines for a number of years as we have built our Dynamical Model for the Growth of Mathematical Understanding. In some what different ways this question has also been on the agenda for other constructivist or interactionist researchers such as Confrey, , Sierpinska, Artigue, Sfard all of whom have addressed CMESG/GCEDM.

[5] Brent Davis in his 1996 book Teaching Mathematics: A Sound Alternative (Garland Press)

Lets look back at our imaginings in terms of these dynamics. As was suggested by the original prompt (which tended to get lost in the later world building) a number of mathematical actions were possible. The physical building of objects; the reflection on both the objects and the building process; re-presentations of the objects in various forms; noticing local (and more general) properties; formulating definitions; relating properties to one another and to properties from other mathematical systems; formulating and proving theorems—these were only some of the actions in which individuals engaged and which one could observe their mathematical knowing. I hope they are visible in your imaginings.

But the materials and the settings themselves as well as the work, talk, questions, re-presentations and properties of others all were fully implicated in any individual's knowing actions. Of course some of this interaction was "competitive": Whose idea was "right"? Who could verify some conjecture first? But like mathematical "competition" everywhere, this competition had a peculiar feature. Rather than being defeated by the actions or words of another, persons here were occasioned by those discovering or explaining or verifying actions of others. They were likely to adopt the idea of the other but in so doing they likely engaged in some new thinking and re-presenting and world changing of their own. These are but some aspects of the inter-actional dynamics that might be noted in interpreting the lived mathematical knowing in the imagining above.

Because the persons involved in the story were members of CMESG/GCEDM, the historic and contemporary practices of mathematics were at the very heart of what was going on in the imaginings above. While it is easy to identify the mathematical cultural impact on these actions of persons with long histories and involvements in mathematics and to see how they in their actions are part of or captured by the "body of mathematics", these impacts could also be studied in the actions of pre-service teachers or children engaging with the prompt and materials that Ralph brought us.

What I am suggesting by making this interpretation of Imagining 1 is that as researcher/teachers or teacher/researchers we can think in this triply embodied way when we try to interpret the mathematical activity of our own students, when we build settings for them or even make presentations to them, and when we interact with them and witness their interactions.

The mathematical knowing in my imagining above is well seen as bringing forth a mathematical world with others. People moved in and out of the activity, brought personal and mathematical cultural things to it and were changed by their participation in it. But what of more "mundane" mathematical activity in schools? Can it be interpreted in this way? To think about that we turn to another imagining—one based on work that I engaged in with Elaine Simmt, Joyce Mgombelo of the University of Alberta and Bob Frizzell, the classroom teacher with a class of 15 to 18 year old high school students in a school in Edmonton with poor performance histories in mathematics.

IMAGININGS 2: NOTICING POLYNOMIAL DISTINCTIONS

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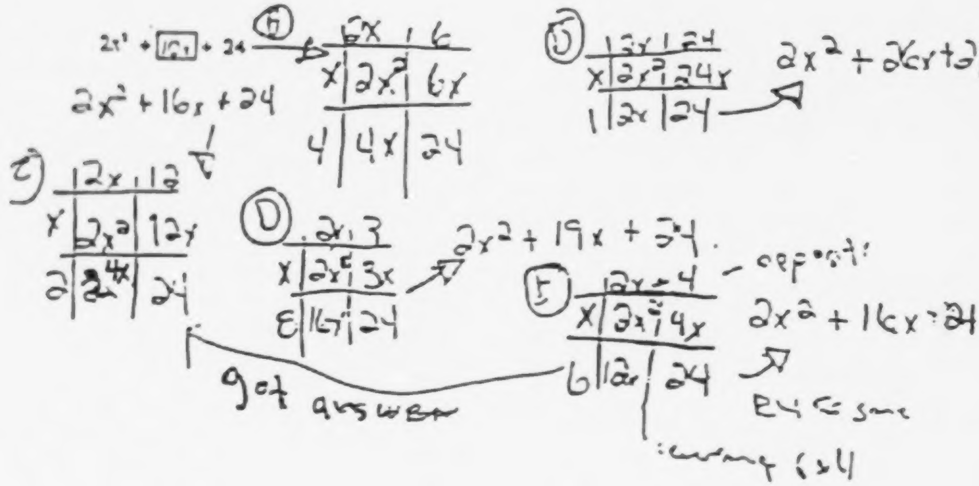
discusses this phenomenon by name. He has presented this explicit idea in our forum as have many others at least implicitly such as Coleman, Agassi, Schatschneider and Henderson.

Imagine a class of 25 high school students who previously have done poorly in mathematics. Some are even taking this course for the third time in hopes of getting a mathematics credit towards a high school diploma. In a special intervention in polynomial algebra the students have been using special polynomial tiles and blocks as well as particular schemes for re-presenting certain polynomial phenomena. For example with respect to multiplying and factoring polynomials they have engaged in such actions as building rectangular polynomial tilings; sketching such rectangles; identifying geometric elements of such sketches in algebraic terms; using multiplication grids (see figure following); using expressions using standard notation; as well as acting to inter-relate the results of some of these acts.



Now imagine two of these students who are working in class. Donny, one of the two, claims to "suck at math". Jen suggests that she has always previously been in "LD" classes. They are working on the following variable entry prompt devised by Elaine Simmt:

> *The following polynomial is required to form a rectangular design as ordered by a client who wants a rectangular tiling. However you cannot read the coefficient for the x term. Before you can pass on the order to your client you need to find out what the missing term might be. Offer a possibility or a list of possibilities to go over with your partner.*
> $2x^2 + ?????x + 24$

Imagine that Donny takes out the tiles and isolates two "x^2" tiles and starts to count out 24 unit squares. In this act and that of picking up some "x" tiles he notes to himself that it will be way too complicated and messy to try to use tiles to find rectangular polynomials that fit. In the meantime Jen has translated the request for polynomial rectangles into a search for items in a list of factorable polynomials. She is using the grid scheme and has found two such polynomials—the first two polynomials in the figure below. Donny looks over at Jen's work and abandons his physical modeling. In fact you can imagine that he looks at the details of the results and starts suggesting other factorings that *they* might test based on factors of 24 as they occur to him. In each case Jen creates the grid, performs the computations necessary, identifies the polynomial and prepares to move on to another possibility. By this time a list of four or five polynomials has been created, the teacher is looking in on what they are doing. Donny is asking what happens when you for example create the "reverse" of (x+4)(2x+6) [B in the figure] by trying (x + 6)(2x +4) [F in the figure]. Donny and Jen were surprised that this "reverse" polynomial was not the same as the original. They are equally surprised and excited by the fact that the polynomial in F did generate the same polynomial as (x + 2)(2x + 12) [C in the figure]. They now start looking for local relationships among factorings. Donny later remarks : " *That actually scares me! Just by putting it in a different part and you get a different answer.*"

To me this is indeed interesting mathematical work especially as generated by supposedly "weak" students. But our point here is not to comment on the quality of the work but to try to interpret it. It is clear that the prompt does not "cause" or determine student behaviours. Each responds on the basis of her or his own histories of action. Even in this brief imagining one can sense that not only is Donny (and for that matter Jen) acting in the setting based on his structure—not only that, but his way of acting mathematically is changing as well with every action he takes even in this short time period.

It is also clear that these students' inter-actions (as well as interaction with the teacher which is only hinted at here) are implicated in their mathematical knowings. For example Jen's work first acts to provide Donny with a potential alternative to working with the tiles. Because of his structure or lived history, Donny is able to take up this alternative and turn it into an occasion for his own actions. Then Jen's carefully labeled re-presentations in her work provides a ground for Donny's seeking and finding relationships. Thus not only are the personal structural dynamics involved in the mathematical knowing, but as was the case in the interpretation of Imaginings 1, the inter-actional dynamics between Donny and Jen and the teacher were fully implicated in that knowing as well. Mathematical knowing in this situation can be observed as doubly embodied.

But what of the embodiment in the body of mathematics? Certainly the cultural tools that were developed in this classroom figure heavily in the practices of both students but particularly for Jen. For example, consider her careful use of grids. This practice is clearly her own, but is traceable to and a part of the mathematical symbolic culture of this classroom.[6] While there is very limited evidence in this imagining, I think we can still ask, " In what ways do the students' actions reflect the practices of big 'M' Mathematics?" The practices of mathematics are best observed in Donny's deliberate attempts to notice relationships and in his emotional reactions to finding them. At least in that sense these students were embodied in the body of mathematics in their knowing action. Any account of mathematical knowing in this setting would be missing an essential ingredient if this cultural component were ignored.

In Imaginings 1 I tried to portray the mathematical actions of members of our organisation—persons with significant and successful histories of working with mathematics and its tools—by pointing to the complicit effects of personal mathematical structures; of inter-actions among them; and of historic and contemporary mathematical practices observable in those actions. In turning to Imaginings 2, I simply wanted to point out that the mathematical actions of persons whose own mathematical histories were not rich and certainly not laden with success also could be considered as triply embodied as well. I turn now to a more speculative venture by looking at CMESG/GCEDM through this lens as well.

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[6] It could be argued that the use of the grid mechanism enabled these students to engage in mathematics that might otherwise been unavailable to them. But by the same token, the use of grids may have inhibited their use of other notations such as standard algebraic expressions and identities. Thus the particular choice of cultural tools at once promotes and limits the mathematical actions of students. Elaine Simmt and I are currently working on this issue with data from the study from which Imagining 2 is drawn and another study conducted in a regular Grade 9 class studying algebra.

CMESG/GCEDM IN ACTION: TRIPLY EMBODIED?[7]

Although the term would be unusual when applied to an organisation, I do not think it a stretch to think of the functioning of CMESG/GCEDM as "embodied". The heart of its meetings are the working groups. By design and by operation such groups are not about the presentation of abstracted disembodied knowledge. No, they are always sites for mathematical actions and inter-actions. Similarly our plenary speakers are not selected simply for their abstracted disembodied ideas found in journals. No, we insist on developing living and embodied relationships with these scholars. We ask them to participate in mathematical actions with us and ask that their words be thought of as generating the possibilities for inter-actions with us. Thus by design, the doubly embodied natures of our engagement with the plenary speakers and our working groups are apparent to me.

A review of the topics of the working groups over the years and the titles of our plenary addresses reveal, at least at a nominal level, embodiment in the body of mathematics as well. That is, it is apparent that the planners of our meetings have deliberately sought to have brought before us for our study both the mathematical practices of contemporary mathematics curriculums at many levels but also the practices of historic and contemporary mathematics (and, of course, including the practices of mathematics education, and other related disciplines as well).

In its deliberate diversity our programme and especially the selection of working group topics has worked to maintain the triply embodied flavour to which I am pointing. A review of the titles of the groups, and I would venture interviews with participants as well, shows a deliberate attempt to develop groups which focus on the personal mechanisms of knowing mathematics - that is on personal structural dynamics. There are also working groups which focus on the role of inter-actions and how these relate to mathematical knowing. There are groups which focus on communication and mathematics. The focus in such groups is not simply on communicative acts which re-present one's mathematical thing/knowing for oneself, but on the roles that inter-personal and ecological concerns play in mathematical action and inter-action. Such sessions fit well with the idea of inter-personal, social and ecological dynamics. Finally, there always have been sessions that deal with mathematics per se. To the extent that I have participated in such groups, I have found their focus to be on practices in mathematics and how such practices impinged on mathematical knowing and the curriculum. While any of these groups might nominally focus on one of the three embodiments, all three—the structural, the inter-actional and the cultural—necessarily figure in the actions of the group.

And what of the ambiance of our meetings? What does attendance at one of our meetings imply? One may "gain knowledge"—as in a list of disembodied ideas or even sources for finding such ideas—by participating. Or one may be able to present ones well developed, if disembodied ideas, in a friendly forum. But I do not think that either of these activities captures what this organisation is nor what participation in it implies. As I tried to illustrate in Imaginings 1, I believe that participation in our meetings is a *bringing forth of a world of mathematical educational significance with others in a truly*

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[7] Although I am sure that he would not think of the organisation in these terms, as I write this section I see it as a tribute to the imagination of David Wheeler who provided much of the grounding for our organisation and who for many years insisted that in our planning we maintain the features that would occasion our meetings to exhibit a unique and what I am here calling an embodied nature.

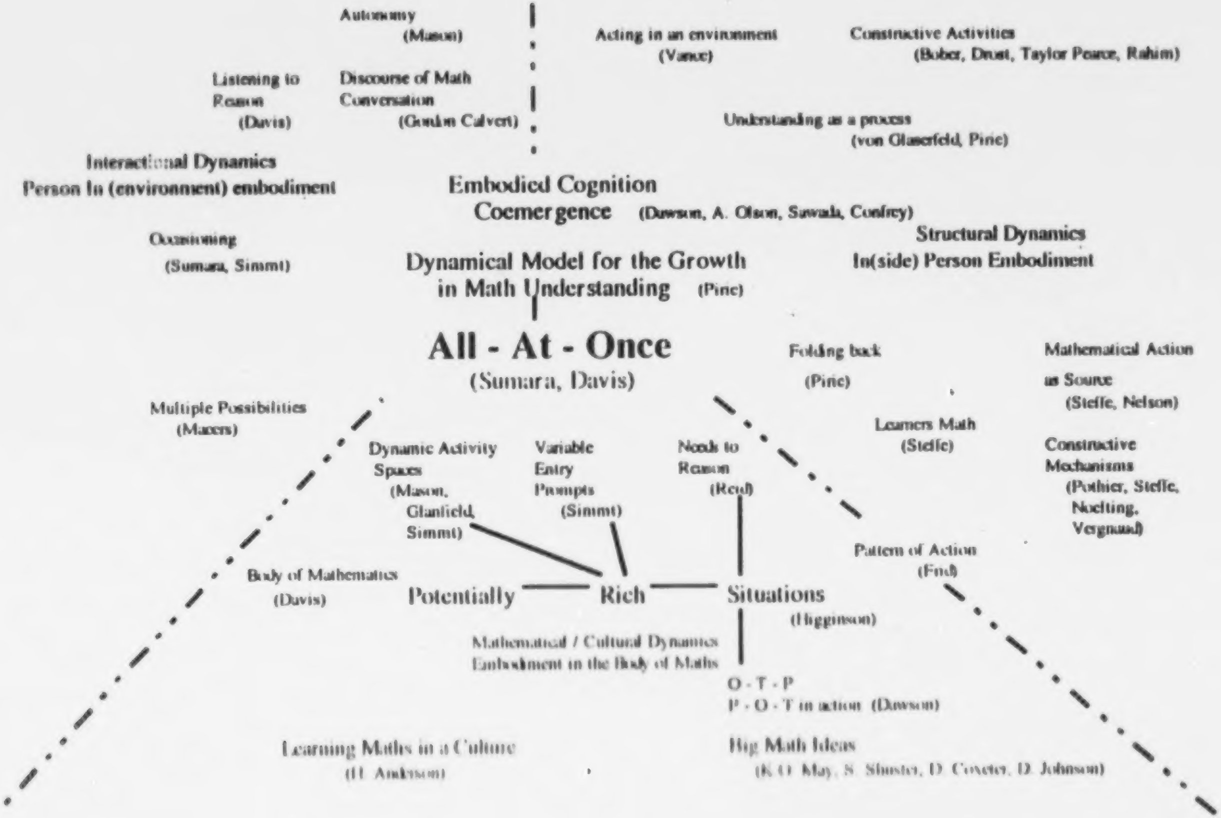
potentially rich sphere of possibilities. Such participation is a way of being in the world. Such triply embodied knowing is a way of living at least for a few days a year.

POST SCRIPT: IMAGINING A GENESIS OF IMAGINING EMBODIED KNOWING

The writing of this signals some 20 years in this organisation, over 30 years of working in mathematics education in Canada, nearly 40 years of working as a teacher of mathematics at least in some way, and well over 50 years of being a student of mathematics. The lived ideas—only hinted at above—come out of living in all of those contexts. While much of the influence of those livings is indirect and much of that is such that I am not conscious of it, I find it useful to provide at least a trace of the genesis of the view that I have sketched out above. On the following page you will find a diagram of that trace. This trace suffers on two counts. First it is necessarily static. But the pathway of the growth of my understanding of embodied knowing is dynamic—it has and is changing in ways that cannot be captured in the diagram. Further, the necessary inter-relationships and complicities among "cells" in the diagram are not apparent either, except for a few meager attempts to connect a few of them. Secondly this trace is incomplete in many ways. But the way I want to highlight it here is personal. That is, this trace is a necessarily incomplete re-presentation of my reflections on living with a large number of people who have influenced my mathematical educational life and are continuing to influence it as I write this. This trace does not reflect all that living and more importantly does not reflect all of the persons and ideas that have served as wonderful occasions for my mathematics education living. There are likely very significant omissions. Perhaps even more importantly, the influences of persons and ideas that are mentioned is understated in this trace.

For the most part I will leave this trace as an artifact for you to think about as you will. It is certainly beyond the scope of this commentary to write about all of the cells, their connections and the intended dynamics of the lived history so re-presented. Even were it not , I have already indulged myself long enough in this essay. Still a few comments are in order.

My sense of embodied knowing really had its start for me in that portion of the diagram labeled the Body of Mathematics. The name Hjalmer Anderson is that of a remarkable high school teacher who created a very special community for doing mathematics in the small mining community in which I grew up. The names May, Shuster and Coxeter are representative of teachers and writers who exposed me to big ideas. Rather early in my career as a graduate advisor I worked with Dawson and Higginson who really stretched my views on such ideas as well as the sphere of my embodiment in the *body of mathematics*. The second phase of my interest in embodied mathematics could be called my constructivist phase. Here I have to add my special tribute to the contributions of Doyal Nelson my former colleague—he pushed me to focus on and interpret the work children engaged in *mathematical action*. This work was refined by long intellectual friendship with Les Steffe whose work and life have occasioned much of the constructivist work that I did on fraction knowing and who introduced me to Ernst vonGlasersfeld and his work. As can be seen at the top of the diagram a central concern which grew out of my interest in how students came to know mathematics, essentially a constructivist concern, is labeled the Dynamical Model and represents work of the last ten years done with Susan Pirie. A central feature of this work is its *operational nature—that is we saw understanding not as an acquisition, but a dynamical embodied process*. Under Susan's influence, our work turned to the implications of interactions especially with the teacher in the changing understanding of the student. At the same time I was reading the work of Maturana and Varela and was in interaction with the likes of Al Olson, Daiyo Sawada and Sandy Dawson about these writings. It was under these influences and based on my history of experiences with *potentially rich situations, constructive mechanisms* and *dynamical understanding* that the notion of mathematical cognition as coemergent took root. Thus, I was enticed to look beyond the mechanisms that an individual used in engaging in mathematical action to try to understand the ways in which the environment and others in it occasioned such actions. The word coemergence itself signals



Autonomy
(Mason)

Listening to
Reason
(Davis)

Discourse of Math
Conversation
(Gordon Calvert)

Acting in an environment
(Vance)

Constructive Activities
(Bober, Drost, Taylor Pearce, Rahim)

Understanding as a process
(von Glaserfeld, Pirie)

Interactional Dynamics
Person In (environment) embodiment

Embodied Cognition
Coemergence (Dawson, A. Olson, Sawada, Confrey)

Occasioning
(Sumara, Simmt)

Structural Dynamics
In(side) Person Embodiment

Dynamical Model for the Growth
in Math Understanding (Pirie)

All - At - Once
(Sumara, Davis)

Folding back
(Pirie)

Mathematical Action
as Source
(Steffe, Nelson)

Multiple Possibilities
(Maeers)

Learners Math
(Steffe)

Constructive
Mechanisms
(Pothier, Steffe,
Noelting,
Vergnaud)

Dynamic Activity
Spaces
(Mason,
Glanfield,
Simmt)

Variable
Entry
Prompts
(Simmt)

Needs to
Reason
(Reid)

Pattern of Action
(Frid)

Body of Mathematics
(Davis)

Potentially — Rich — Situations
(Higginson)

Mathematical / Cultural Dynamics
Embodiment in the Body of Maths

O - T - P
P - O - T in action (Dawson)

Learning Maths in a Culture
(H. Anderson)

Big Math Ideas
(K.O. May, S. Shuster, D. Coxeter, D. Johnson)

more than 10 years of work which I have discussed elsewhere in more detail. Since 1990 I have been working with several colleagues especially Brent Davis and Dennis Sumara on elaborating Maturana's essentially biological ideas in educational term. As mentioned above it is Brent's notion of Body of Mathematics which served as a take off point for me as I created a circle in my own intellectual history by connecting my other theoretical and research interests back to my original interests in the body of mathematics. I have named this new but old theoretical, research and pedagogical enterprise the study of *mathematics knowing as triply embodied*. The names of a large number of graduate students pepper this trace, particularly as it re-presents this recent enterprise. They have figured heavily both in my work and in my thinking. Although I haven't pointed out all of their influences here nor those of other colleagues mentioned or unmentioned in the trace, my *imagining of embodied cognition* would have been fundamentally constrained or perhaps non-existent without them.

WORKING GROUPS

Working Group A

AWARENESS AND EXPRESSION OF GENERALITY IN TEACHING MATHEMATICS

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ABSTRACT

> Pursued in the spirit of a philosopher and not of a shopkeeper, arithmetic has a very great and elevating effect, rebelling against the introduction of visible or tangible objects into the argument. (Plato, *The Republic*)

> We must distinguish (if we can) between new ideas that have come to stay and new ideas that arise from chance and change—ideas which are in the fashion, and have in consequence a certain air of smartness, but come off badly when subjected to the wear and tear of the classroom. The worst of it is, when one idea gets into fashion it pushes another idea out of fashion. And the other is often the better of the two. [Ballard 1928, preface]

Over time, generality has been expressed in different ways. The principal influences, such as Babylonian and Egyptian scribes, Diophantos, medieval abacists, Cardano, and Viète, to name only a few, each offered a form of generality, but the implicit and explicit ways in which they expressed generality is also part of what generality is, or was, for them. Authors of arithmetic and algebra teaching textbooks from the 15th century to the present have similarly reflected differing awarenesses of generality. By looking at how selected authors have worked in the past, participants will be urged to reflect on their awareness of generality when 'doing examples' and enunciating theorems, and how they explicitly (or implicitly) indicate generality in their teaching.

OUTLINE OF ACTIVITY

The approach adopted was to stimulate current 'taken-as-shared' experience with the intention of triggering access to similar experiences in the past, in order to provide a rich supply of examples of generality and the expression of generality to feed observation and discussion.

In each session, participants were invited to undertake one or more tasks, drawn from draft notes prepared in advance (Mason 1997). Usually this involved work on one or more mathematical problem, but sometimes it was more concerned with style of presentation of a mathematical argument. Participants worked on the task individually or in concert as they felt moved, followed by a short period of reflection and reporting to the whole group on observations and emerging issues. At the end of the session there was further reflection, and an invitation to make notes. The following session began with further observations and discussions, in the expectation that what had remained with a person was likely to be of ongoing significance. Our role as leaders was not to obtain consensus but to provide stimuli for individuals and groups to bring to expression conjectures and observations for further consideration.

Comments and observations by participaNts have been integrated into the report, except in the last section where final reflections are listed separately.

DAY 1A

The opening task was a silent presentation of the following sequence taken from Peter Taylor's draft book distributed at the meeting:

$$2\sqrt{\frac{2}{3}} = \sqrt{2\frac{2}{3}}$$

$$3\sqrt{\frac{3}{8}} = \sqrt{3\frac{3}{8}}$$

$$4\sqrt{\frac{4}{15}} = \sqrt{4\frac{4}{15}}$$

It was suggested that everyone present knew (or expected) what would be written next. It was observed that a sense of what was coming next was not the same as a sense of form, which was different again from having a sense of a formula. In other words, one can have a sense of nextness without a sense of individual pattern, a sense of pattern without an explicit expression of that pattern, and expression of pattern may require several versions before the expresser is satisfied with it as an expression of perceived generality. Furthermore, expressing perceived generality often leads to awareness of further possible generality. Thus the initial task set the scene as intended, for the larger issue is whether one can detect awareness of generality in the way in which arithmetic and algebra are presented in textbooks through the ages.

The first main task was a choice between the following.

Initial Task:
To express any fraction as the sum of distinct unit fractions. To express any fraction as the sum of two other fractions whose numerators are given.

To express any fraction as the sum of two other fractions whose numerators are given.

Suggestion:

First, work at these yourself. For a real challenge, try using words but not symbols! When you have worked at these for yourself, try expressing any result you have in words, as a rule. Then consider the following rules given in the *Ganita-Sara-Sangraha* of *Mahaviracharya* (c.830 AD), and consider what you are called upon to do in order to make sense of them.

> "The denominator (of the given fraction) when combined with an optionally chosen number and then divided by the numerator so as to leave no remainder, becomes the denominator of the first numerator (which is one); the optionally chosen quantity when divided by this and by the denominator of the sum is the remainder. To this remainder the same process is applied." [Datta & Singh 1935 p201]

> "Either numerator multiplied by a chosen number, then combined with the other numerator, then divided by the numerator of the sum so as to leave no remainder, and then divided by the chosen number and multiplied by the denominator of the sum gives rise to one denominator. The denominator corresponding to the other (numerator), however, is this (denominator) multiplied by the chosen quantity." [op cit. p202]

Alternative Initial Task:

A good runner walks 100 paces (*a*), a bad runner goes 60 paces (or *b*). Now the latter goes 100 paces (or *c*) in advance of the former, who then pursues the other. In how many paces will they come together?

A hare runs 100 paces (*a*) ahead of a dog. The latter pursues the former for 250 paces (*b*), when the two are 30 paces (*c*) apart. In how many further paces will the dog overtake the hare?

Suggestion:

First, work at these problems yourself. They are taken from the *Chiu-chang Suan-shu* (Arithmetic in Nine Sections, at least as early as 500 BC) [Mikami 1913 p16]. For a real challenge, try using only words, not symbols. Express in words as rules how to solve other problems of 'that type'. What constitutes 'type'? Perhaps construct your own variants.

The original text may not have inserted symbols to suggest generality as Mikami has done, but the solution would have been given in words, using names for the quantities the numbers particularise.

In the event, all but one group chose the first task. It proved to be rich in revealing different approaches and different ways of conceiving the expression of an algorithm. It also required some sort of validation or proof, whereas the second task was more straight forward and did not seem to require 'proof'. No-one who worked on the first task got as far as examining the Indian solution.

Observations

One group developed and shared an algorithm without ever explicitly stating it, through sharing considerable mathematical expertise and experience. This might be precisely the state of a group of scribes who shared specific examples but never actually articulated the generality. It might also be the intention of an author offering chosen examples to students that the students develop a general algorithm through contemplation of particular examples. Other groups worked at expressing a general procedure and then deciding if it would terminate.

For some, starting to generalise introduced tension while staying with the particular was comfortable; for others it was all too easy to try to out-generalise others. Generalising too quickly, and sometimes even at all, can prove futile when the symbols become too hard to handle, or when they fail to capture the essence of the problem. This tension is present in most classrooms and lecture halls too.

One group started with numbers and reached for generalisation, while another started immediately with symbols. For some, numbers were particular, for others, generic and the same was true for symbols: for some groups which used symbols quickly, those symbols were at first only of very limited generality, while for others, the symbols were an expression of perceived generality. It is only when the need to prove a conjecture arises that the symbols were seen as essential to the generality of the argument. What is important is that you start with confidently manipulable entities and use them to try to get a sense of what is going on.

Questions arose as to what constituted a suitable generality. Generality involves both the statement of something as invariant and a statement of what is permitted to change, and over what domain. Often the domain of change is overlooked or understressed, as in geometrical statements which often do not emphasise or even omit altogether the domain of changeability: 'for all triangles...' or 'for all cyclic quadrilaterals ...'). Instead of stating the domain, one can take a Lakatosian perspective (Lakatos 1976) by telling students that some objects (e.g., triangles) satisfy such and such a property; and asking them to find those which do not satisfy it. This opens up possibilities for the natural propensity for finding counter-examples and for exploring boundaries (Michener 1978; Dyrslag 1978).

People working on the hound-and-hare problems found themselves having to make assumptions which made the problems tractable, and so trying to enter the culture of the setter and perhaps the culture of an activity of the time. This demonstrates one of the potentials for work on word-problems from different periods: stimulation to discuss cultural differences. A further difference with the first task was that the two problems posed did not generate a direction for investigation. The group members got stuck on seeking similarities and differences between the problems, and so did not explore common generalities. Generality was clouded by the apparently practical or concrete nature of the problem contexts.

In the first task, some who had gone for symbols on the fractions had written immediately that $\frac{1}{N}+\frac{1}{N+1}=\frac{sum}{product}$ and found it useful to have names for chunks. At times people talked geometrically, or at least employed a spatial metaphor: 'take out the largest $1/N$'. Some reported not being able to follow details of what others were saying, but construed a general sense.

An aim of these tasks was to generate an encounter with generality and with the challenge to express that generality. Interpreting other people's generality, such as the Indian 'solution' exposes the difficulty in expressing generality in words as a pedagogic device. The importance of recognising generality is no modern idea, as the following extract suggests. In response to a student who, having been told that knowing how to calculate was all that was required if he applied his thoughts appropriately, returned to ask for further instruction, Master Chén said in the Zhoubi Suanjing (first century BC):

> That is because you are not familiar with your own thought. ... but you have still not got things clear, that is to say, you still cannot generalize what you have learnt. The method of calculation is very simple to explain, but it is of wide application. This is

> because 'man has a wisdom of analogy' that is to say, after understanding a particular line of argument one can infer various kinds of similar reasoning, or in other words, by asking one question one can reach ten thousand things. When one can draw inferences about other cases from one instance and one is able to generalize, then one can say that one really knows how to calculate. The method of calculation is therefore a sort of wisdom in learning . . . The method of learning: after you have learnt something, beware that what you have learnt is not wide and after you have learnt widely, beware that you have not specialized enough. After specializing you should worry lest you do not have the ability to generalize. So by having people learn similar things and observe similar situations one can find out who is intelligent and who is not. To be able to deduce and then to generalize, that is the mark of an intelligent man . . . If you cannot generalize you have not learnt well enough . . .. [quoted in Li and Shíràn, 1987 p28].

Conjectures

The conjecture was offered (supported as it turns out by Gillings (1972) and by Hoyrup (1990) that Egyptian and Babylonian scribes might have been well aware of generality but chose only to present particular examples, which are often posed as questions, perhaps as a pretext for showing off the intellectual skill of the scribe. Perhaps it was through experience of having tried to express generality in words and finding it ineffective and lengthy that scribes knew not to waste their precious scribing energies. Perhaps it was assumed that students would be in the presence of a scribe who could read generality in the particularity and so provide students with forms to follow and access to generality. It is reasonable to assume therefore that how and when to express generality to students (not colleagues) has been an issue since before written records.

Participants were invited to consider a number of conjectures concerning the appearance of implicit and explicit generalisations in historical texts, with a view to considering the same issues in their own teaching. Of particular interest are the transitions which students are called upon to make, and ways in which teachers can support and provoke those transitions. In school, technique is the focus of each topic: technique is examinable, and provides a procedure for resolving a class of problems. But are students aware of the class of problems it resolves? Could they express this generality?

The historical role played by word or story problems in the development and teaching of algebra is complex and intricate. Word problems were initially arithmetic in nature, but people soon found that algebra often helped in the resolution of more difficult ones, and that instead of resolving a particular problem only, one could formulate (literally) and resolve a generalised version (as suggested for example in the way Mikami offers letters with numbers in the second initial task above). For example, Diophantos posed number problems in general but solved them in the (presumably generic) particular; Newton posed and solved each problem in general symbols, and then offered a particular example. But historically, algebra texts rapidly turned into textbooks on the 'arithmetic of polynomials', and word problems remained in the domain of practical arithmetic texts, reappearing in recreational texts from the 17th century and sporadically in algebra texts, particularly in the early 1900s. Word problems offer a rich cultural and pedagogic means for stimulating students to perceive and express generality, in preparation for later algebra (Polya 1962; Swetz 1987; Gerofsky 1996, 1997).

DAY 1B

On return from coffee break, participants were directed to the following

Babylonia

7: I found a stone but did not weigh it; after I added one-seventh and added one-eleventh, I weighed it: 1 *ma-na*. What was the original weight of the stone?
The original weight of the stone was 2/3 *ma-na*, 8 *gin*, and 22 1/2 *se*.

9: I found a stone, but did not weigh it; after I subtracted one-seventh, added one-eleventh, and subtracted one thirteenth, I weighed it: 1 *ma-na*. What was the original weight of the stone?

The original weight of the stone was 1 *ma-na*, 9 1/2 *gin*, and 2 1/2 *se*.

$$\left(x-\frac{x}{7}\right)+\frac{1}{11}\left(x-\frac{x}{7}\right)-\frac{1}{13}\left(\left(x-\frac{x}{7}\right)+\frac{1}{11}\left(x-\frac{x}{7}\right)\right)=1{,}0$$

$x = 1{,}9;30{,}50$ *gin* [Extracts from Tablet R. YBC 4652 p. 100]

Bearing in mind that Babylonians used base 60, work out the meaning of *ma-na*, *gin*, and *se*. Is there any mathematical or pedagogical consequence of the choice of numbers?

Various authors have put forward the notion that Babylonian scribes knew general methods, and that the tablets we have are examples of those methods being applied. Some seem to be the workings of students; others could be examples for students to follow. This raises the deep and eternal pedagogic question of the role of generality and of particularity in teaching students. Students are driven by assessment, which,

if problem-oriented, focuses student attention on methods and classes of problems, and may divert them from appreciating underlying structure;

if theory oriented, may focus student attention on memorising theorems and proofs;

In the spirit of Babylonian mathematics, in which fractions are always written as sums of reciprocals apart from 2/3, describe in words how to 'do' an Unweighed-Stone problem, and describe the class of problems which your general method will solve.

Unfortunately people got bogged down in trying to reconstruct the units, and so did not always focus attention on the implied (pedagogic) generality. It is interesting to note that the participants did not attempt to produce a single generalisation despite the fact that there were two problems as in the previous task. The unfamiliarity of the context appeared to inhibit completely any desire to generalise. This state is shared by students: unfamiliar contexts can stifle the freedom needed to consider generality.

Reflection

In retrospect we wish we had instead drawn attention to the succeeding page of the notes taken from the Rhind Papyrus in which generality is even closer to the surface:

Problems 24-27 of Rhind Mathematical Papyrus

A quantity and its $\bar{7}$ added becomes 19. What is the quantity?

A quantity and its $\bar{2}$ added becomes 16. What is the quantity?

> A quantity and its $\bar{4}$ added becomes 15. What is the quantity?
>
> A quantity and its $\bar{5}$ added becomes 21. What is the quantity? [Gillings, p. 154]
>
> Problem 28 of the Rhind Mathematical Papyrus (1650 BC) by the scribe Ahmes (1620? BC - 1680? BC): [Think of a number]. Two-thirds is to be added. One-third [of that] is to be subtracted. There remains 10.
>
> Make $\bar{1}\,\bar{0}$ of this, there becomes 1. The remainder is 9. $\bar{\bar{3}}$ of it, namely 6 is added. The total is 15. $\bar{3}$ of this is 5. Lo! 5 is that which goes out, and the remainder is 10. The doing as it occurs!

In modern terms

> Think of a number, add two-thirds of it. Take one-third of that away. What remains?
>
> From your remainder I take one-tenth: that is the number with which you started. [Followed by a check of this] That is how it is done! [Gillings, p.182]

Note that Gillings has introduced a generality, reading the given answer 10 as generic, and converting a specific problem into a think-of-a-number game.

The observations which followed work on the Babylonian task indicated that participants had come up against the historian's problem of interpreting what an author meant, by making use of the solution to reconstruct the problem's intent. We also discovered that in reading historical texts it is easy to make unwarranted assumptions, such as that a word for a unit of measure always has the same meaning!

DAY 2A

We began by reviewing issues which had arisen the first day (summarised above). Participants then broke into groups to consider either a collection of problems related to Diophantos' first problem, or to consider Chuquet's courier problems and try to construct a generalised courier problem.

Diophantos (200?-284?) and His First Problem

van der Waerden (1983) finds that when Diophantos solves determinate problems, he uses methods that are identical to those used by the Babylonians, though Diophantos uses a symbol for the unknown (examples of which are found in an Egyptian papyrus pre-Diophantos *op cit.* p. 104, p. 108-109) and so leads the reader to a solution, whereas the Babylonians only recorded the method-formula. But Diophantos also introduced indeterminate equations, and developed clever methods of resolving these, with the use of only one unknown.

Thomas Heath (1861-1940) translated some of Diophantos as well as Euclid (Heath 1885). The 130 problems are solely about finding numbers, and just in the first of the six extant books (there were 13 originally), the problems range from

1. To divide a given number into two having a given difference. (e.g., given 100 and difference of 40)

through

32. To find two numbers in a given ratio and such that the sum of their squares also has to their difference a given ratio. (e.g., given 3:1 and 10:1)

to

39. Given two numbers, to find a third such that the sums of the several pairs multiplied by the corresponding third number give three numbers in arithmetic progression (e.g., given 3 and 5).

Generality is implied in the verbal statements of the problems but his resolutions were specific; later Arab sources employed symbolic methods to resolve them generally. Diophantos' first few problems appear in most problem collections and textbooks, but few authors draw explicit attention to the fact that many apparently different problems can be seen as contexts for Diophantos' first problem:

What is it that enables you to recognise some of the following as variants of Diophantos' first problem, or extensions? What does structural awareness consist of?

> A bottle and a cork cost 24*d*. The bottle cost 2*d*. more than the cork. What was the price of the cork? [Ballard 1928, Chapter XX]
>
> A man rows down the stream at the rate of 6 miles an hour, and against the stream at the rate of 3 miles an hour. What is the rate of the stream? [Ballard 1928, Chapter XX]
>
> A man pays with a $5, (*d*), note two bills, one of which is six-sevenths, (*a/c*), of the other, and receives back in change seven, (*h*), times the difference of the bills. Find their amounts.(p.236) [Anonymous text from Waterloo library].
>
> A boy, being asked his age and that of his sister, replied, "If I were 3, (*a*), years older, I would be 3, (*n*) times as old as my sister; but, if she were 2, (*b*), years older, she would be one-half, (1/*e*), as old as I am." How old was each ? (p.236) [Anonymous text from Waterloo library]

Thought in retrospect: It seems to be more effective to offer participants a set of problems and ask them to sort them in some way (before actually solving them), and only then to work at solving them and reclassifying them, than it is to offer the task as given here. In previous workshops, sorting first and then solving later brought a subsequent awareness to the fore that underneath the context, the problems have the same structure. In our case, it turned out that participants who were used to immediately expressing Diophantos' problems in symbols were prevented from pushing deeper into the question of locating what, in the following problems, blocks emergence of an awareness of commonality between problems. The relatively recent emphasis on how to read and interpret word problems has actually blocked access to underlying generality. This phenomenon will also play a role for those who attempt the following problems.

Chuquet's Courier

22. Two men depart on one day and in one hour, that is to say, one leaves Paris to go to Lyon which is a hundred leagues by road, and the other man leaves Lyon to go to Paris and makes the journey in 7 days. And the one who is going from Paris to Lyon makes the journey in 9 days. To determine after how many days they will meet each other. [Flegg et al. note that Chuquet gets the answer wrong, but gets other similar ones correct.] (Flegg et al. 1985, p. 204)

Problems 98-100 and 103-108 illustrate the case where one or both men travel not at a constant rate, but increasing their daily journey in arithmetic progression, while the other problems between 125 and 128 introduce the implausible complication of travelling backwards at night, familiar to most readers in the guise of the 'frog in the well'. (p. 204) [But it might be noted, often the answer given fails to take into account what happens on the last night!]

> If a log starts from the source of a river on Friday, and floats 80 miles down the stream during the day, but comes back 40 miles during the night with the return tide, on what day of the week will it reach the mouth of the river, which is 300 miles long? [*Ans*: Friday] (Jones 1912, p. 10)

What variations might Chuquet have included? Is there an all encompassing generality for courier problems?

An appendix in the notes offered a range of courier problems, with the intention that participants might later see whether their generality encompassed the variations offered in various later texts. It was our deliberate intention to obscure the boundary between variation, extension, and generalisation.

The Courier group soon went well beyond the canon of the problem, by thinking geometrically. Since the courier problem can be resolved by drawing a graph and locating where two straight lines cross, this approach generalises to setting such as:

> if two people set out from different places and or at different times to undertake continuous journeys such that eventually their paths will cross, find the crossing place or time.

This places the problem in the domain of functional analysis, but perhaps loses some of the attraction of courier-based variants. It does show up Hare-and-Hound problems as courier problems. The form $\frac{d_1}{d_2} = \frac{v_1}{v_2}$ permits the insertion of courier-type data, and so in some sense represents a generalisation. Certainly it provides symbolic expression of underlying structure, though whether you need to appreciate the structure in some deeper way in order to decode and recognise it in a fresh context is another matter.

Because we are already familiar with algebra, it is difficult to experience what it might have been like without algebra to pose and resolve some of these questions. Certainly the extensive and varied attempts by text authors in India and in Europe to routinise the use of the *rule of three* and of *false position* suggests that a majority of the population preferred to have a formula rather than to think through the structure for themselves. There is ample evidence that teachers in every generation have resorted to routinising problem methods for their students in the hope that this will enable them to obtain correct answers. Perhaps teachers have always found it difficult to teach students to think or, put another way, to release their powers of thought.

DAY 2B

Looking at Texts and Word Problems

It was suggested that in looking at arithmetic texts, there were three different ways in which generality was invoked:

Generalisation through repetition:

> Students do enough examples to become familiar with doing 'that type of problem'. Of course they have to abstract 'that type of problem', and attend to the technique rather than simply doing each

'example' in turn. Recognising something as an example means it must be an example of something, so there is an element of bootstrapping in this approach. It was popular in many texts, especially in North America in the 18th and 19th century.

Generalisation through particularising a stated generality and re-generalising

One common style in arithmetic books (as early as 15th century if not before) stated a general rule in words first, then offered worked examples. Gradually texts added commentary and observations to these. It became common to then restate the general rule, perhaps in symbols, before (perhaps) offering exercises to attempt.

Generalisation through using a single (or perhaps a few) generic examples

This requires students to be able to read the general in and through the particular, and was what most of us did when working on the Babylonian tasks.

Word problems are disappearing as a genre of mathematical activity, and even where they are still used, they most often appear as challenges at the end of a topic to keep the faster students occupied while the slower ones catch up. One aim of the working group leaders had been to draw attention to the role and value of word problems as a device for stimulating and supporting the expression of generality. Support for this can be found from Polya:

> I hope that I shall shock a few people by asserting that the most important single task of mathematical instruction in the secondary schools is to teach the setting up of equations to solve word problems. (Polya 1962, vol. I p. 59)

Polya must have had in mind fostering thinking, not routine application of technique. Yet since the 1950's authors and researchers have tried to routinise solution of word problems by offering devices (such as tables) and rules of thumb for translating words into symbols, for the most part with little success. For us the word problem not only offers an opportunity to extract a mathematical question, but the chance to experience the recognition of a 'type' or 'class' of problem, and to express a method for solving anything in that class.

If a student enters an examination with little idea of the types of problems which will appear, they are at the mercy of the examiner and of their own fears; if they enter it able to say of any particular question:

> Is that the best they can do? I can state and solve much harder versions of that silly problem!

they are in a much stronger position. Hence the value in working on typing and classifying problems. One useful technique is to make a collection of cards each with a problem on it, such as might appear in a revision exercise. Asking students to sort the cards in anyway they like usually produces several different approaches. Inviting students to then describe the basis for other people's sorts brings them into contact with other ways of perceiving the questions, and offers them an opportunity to consider a more mathematically effective perspective.

Observations

The notion of generalisation itself becomes ambiguous, especially as it was being used to incorporate variation and extension, not to say abstraction.

Generalisation may appear to be more of an individual matter than a group matter, because each person's expressed generality may have to be reconstructed by each individual through its particularisations, unless it is expressing the insight of all members of the group. When the task becomes a challenge, collaborating strangers are more likely to turn inwards to their own thinking than to stick with the ebb and flow of the group. Certainly sometimes it is necessary for individuals to work for and by themselves; coming back to collective activity afterwards requires disciplined practices.

When one person suddenly sees something they are then in the position of trying to explain or convince others. Effective generalisation in a group requires real co-thinking, so that what is said captures what each member of the group is thinking. It takes effort to develop the working practices which support this collective (not just collaborative) action. Collaborative working produces individual generalisation and the force of the group to check and test conjectures until there is a collaboratively constructed convincing argument.

In the world of teaching mathematics, the dominant ideology disparages repetition, and hence one of the routes to experiencing generality. Instead, generalisation through a single, or perhaps a few generic examples is favoured. Justification for this is based on issues of motivation and representation. Stressing generality emphasises verbal description of the general, and hence supplies natural support for stimulating discussion.

Inviting others to generalise runs the risk that they will move in unforseen directions, as happened in the workshop, and perhaps miss a fruitful but more limited domain of generalisation. Some means is needed for indicating boundaries in a sensible domain of generalisation before students give full rein to their powers of abstraction. The existence of general objects which the activity is intended to bring to attention must be perceivable by the participants. To find the energy to search for something you need confidence that it exists. Generalisations occur to mathematicians , but to experience this, students need to be urged to seek them. The role of counter-examples must also be kept in mind.

When students are given a series of problems, the task may be interpreted in different ways by students:

- as solving problems which are distinct and unrelated;
- as solving problems by application of a previously studied technique;
- as solving problems with the aim of seeing what is common to them and appreciating both the class of problems solvable by a common technique, and a sense of that technique and what it will solve.

DAY 3A

Again, participants were invited to recall and report on issues and observations which had arisen the previous day (reported above).

DAY 3B

Previously we had been looking at the presentation of arithmetic and the use of word problems for introducing algebra. It was time to look at tertiary practice. We each read an extract from Herstein (1964) in which he proves that a commutative ring with unit element but no non-trivial ideals is a field, and an extract from Loomis (1974), in which he illustrates a differentiation of a quotient of two polynomials. Then we discussed our reactions in smaller groups. The focus of attention was the nature and type of generality involved in the extracts. It took most of the session to read individually and then work through one extract.

One group working on Herstein noted that individuals would have found it very difficult to 'get into' the Herstein, despite having had facility with such theorems in the past, but the presence of the group helped focus attention and make sense of the theorem. We noted the many different ways in which generality was indicated, as in 'for any ...', 'every ...', and how in English there is 'any', 'all', 'a', 'every', 'always' whose construal in mathematics requires induction into the *register*. 'Let ...' is often a sign of particularity, but 'Let ... be ...' signals a generality or generic object. It was noted that Herstein's proof contained no indication of boundaries, of why the different conditions are needed. Herstein's text was and is still famous for its interchange between advance organiser, content, and meta-comment.

PLENARY REFLECTIONS

Indented text are observations made by participants, grouped under various headings.

General and Particular

> The idea that struck me most was the value, once a particular solution is achieved, in generalising. It is so vital to teaching and learning mathematics.

> A general statement can guide students to reconstruct it, but it can also freeze them out; it hides the process by which it was obtained (the experience, the scope, what was stressed and what was consequently ignored).

> I was struck by the role of the particular as providing something concrete on which to reason (contributing to positive affect), and creating my own data in which to seek patterns; the role of symbols to organise the data, to make certain features explicit, but that it does not always help.

> Different perspectives might be obtained by asking both 'Does this always work?' and 'What works this way?'. Generalisation seen as invariance amidst change reinforces the notion that boundary examples are important for appreciating the nature and scope of a generalisation.

> Generalisation goes beyond arithmetic–algebra, for it includes the geometric.

In the working group we wanted to stay with the arithmetic-algebra borderline, but of course people found it useful to draw diagrams and graphs. Geometric generalisation has to be 'picked up' because it is not formalised the way it is in the symbols of algebra. Dynamic geometry software has opened up possibilities for physical-virtual manipulation of geometric generality, and it is to be hoped that there will be significant further developments in this direction for other areas of mathematics. It is an interesting question as to what it might mean to undertake geometric manipulation by analogy to algebraic manipulation, with the possibilities now afforded electronically.

> Working in the group helped me 'see' that students move between general and particular all the time; that what is general for some may be particular for others. Thus I was reminded of things I had stopped noticing.

> Are we trying to isolate generalisation and give it a special status? Are we trying to classify types of awareness? If so, why? Are some awarenesses better than others? Do we seek a hierarchy?

Certainly it was the leaders' intention to focus on generalisation, and to give it special status, as being at the heart of appreciating mathematics (with proof being a central theme). Awareness of generalisation, of opportunities for students to try to express generality themselves, is central to effective teaching. As for hierarchies, that was not our intention. It is not that some awarenesses are better than others, but rather that the more one is or can be aware of, the more one notices and the more sensitive one can be to students. Furthermore, if appreciating and understanding mathematics is based on appreciating and expressing (and then manipulating) generality, developing sensitivity to generality is useful to us as teachers. Mathematics proceeds by routinising the problematic; from a constructivist perspective, it is important to problematise the routine in order that students de-problematise for themselves (collectively and individually) by re-constructing the routine. To a mathematician a solved problem is like a broken sword on the battlefield, as the Sufi proverb has it. Whereas a method for solving a class of problems can, it is assumed, be handed to a colleague for their effective use, it cannot usually be successfully handed to a student unless the student is well primed with appropriate past experience and appreciates the problematicity of the problem being solved. To a teacher, a solved problem is a potential stepping stone to extending appreciation of type, of particularity and of generality, and even of formula.

> The art of generalising enables you to see connection, relations, commonalities, etc. that you don't see before. It also helps hold seemingly disparate things together. Whether we generalise from one genuine example (generically) or from a host of examples (which are not actually examples until they are recognised as examples-of something), what aspect we stress is crucial. The fruitfulness of generalisation in mathematics lies in the consequences of that generalisation, and not from generalisation for the sake of it.

> Two possibilities arise in teaching: (a) start with a generic example, or with many examples; (b) start from the general and move to examples.

The most common strategy in textbooks through the ages is to offer a general rule and then a few or many examples, with or without practice exercises, and then a restatement of the general rule. Some authors offer justification either early, or later, in the exposition. Just in working with examples, the presence of a general can help direct students to reconstruct the general for themselves from examples. To treat a single instance as a generic example requires knowing what is generic (changeable) and what is invariant, and this is not always obvious. A useful exercise introduced by Brown and Walter (1983) is to take a statement (such as 'the sum of the angles of a triangle is 180 degrees') and to read it out loud placing extra stress on a single word. The effect of the stressing is to invite asking 'what if that were different—what might it also be?'.

Word Problems

> Are word problems like cartoons in some ways? They refer to typical or recognisable everyday situations but are often exaggerated and unrealistic in detail. What is it about animated cartoons and comics that appeals to children, and in what way might this appeal be invoked in word problems?

Generalisation through the generic

> Are word problems akin to jokes and riddles, even fairy tales? these are forms which through generic examples define culture and its categorisation of the world?

Generalisation through repetition:

Language is learned through enculturation; embedded in language, children live the rules, structures and vocabulary. Can word problems function this way? Can mathematical generalisation be enculturated?

Categorising and Connecting

Pigeon-holing problems by the technique used to resolve them seems a familiar and un-mathematical activity. More important is to make mathematical connections.

Is the categorising of problems a form of generalisation? Is any classification scheme a generalisation?

Some attempts by texts and teachers to classify problems by providing very narrow categories would appear to work against generalization. Often categories of problems (i.e., related rates in calculus) which possess general schemes both for presentation and solution are broken down into subcategories based upon surface context features (i.e., tanks or troughs filling with water). This approach works to improve examination results since questions will always come from one of the identified subcategories, but students are left to construct overarching views of the problems and general solution schema. Research (Reed, 1989) shows that developing general solutions from analogous cases is not an easy task, and may require some direct intervention that identifies parallels and common features. (Goeff Roulet)

Certainly seeing generalisation is a personal matter, however supported by collective discourse and practices. Ownership of generalisation often seems difficult. Yet 'seeing generality through a particular' is a frequent and sudden process, and fundamental to the functioning of and to functioning in language.

Behold

Reference was made to Bhaskara's diagrammatic proof of Pythagoras, with the single word *Behold!* It also appears in the reading of a phrase cited by Gillings (1972, p. 233) which comes at the end of the statement of a problem: *Behold! Does one according to the like for every uneven fraction which may occur.* Examination of the possible etymology (no claims were made or are being made!) suggests an interpretation as *to be held by, to stay with*, that is, to *gaze at* and *see through* the diagram; experience generality indicated by the particular. One is beholden to someone who has done you a favour, and perhaps beholden to an author or teacher who has focused attention in such a way as to enable you to behold. There is also a sense of seeking a particular state, not just glancing and then proceeding without really taking it all in, itself an interesting metaphor, akin to St. Paul's admonition, to read, learn, and inwardly digest.

To behold requires an openness which is not usual. Recent research has suggested that learning can release endorphins in the same way that physical exercise, food, and sex can. This would confirm an ancient theory that seeing generality through particularity (and also seeing particularity in generality) can release energy which can then be transformed into further work or effort. So the role of a teacher is to assist students in using the energies effectively.

Voice

David Pimm mentioned that Euclid (in the translations we have) shows no evidence of a sense of audience. There is no pronoun such as 'we' which might include the reader. Indeed, Euclid's objective, timeless tone is a model for subsequent axiomatisation.

Someone commented that they have always loved geometry but hated Euclid, and wondered if the tone was the source of their experienced disparity. It is often difficult in reading text to distinguish between the voice of the author as individual, the voice of the author as representative of a community of experts, and the voice of the author trying to encompass the reader in a collective 'we'.

Presentation

When contemplating the problems on p. 25: Are these unconnected or is there some subtlety in their presence, even in their order? This is one of the issues in texts from all periods, including today's. There was an extract from Carr (1886) in the notes which was reputed to be an inspiration for Ramanujan. Carr provides another example of how it is easy to see a list as a list, but possible to detect underlying structure and hence a suggestion of further unstated or implicit possibilities. How might students be supported in locating and appreciating underlying structure, in *working on* exercises rather than merely *working through* them?

Teaching

Is there something to be gained from relating teaching to the telling of parables?

Should into Could

There was a strong element of *should* in observations: Should one ... ?; Should we ...?; We should ... . In working with teachers, many of whom similarly use *should*, it has proved liberating to suggest replacing *should* by *could*, transforming the moral imperative into a possibility to be considered. There is no one perfect or best way to teach, or to act in a classroom; there are only possibilities. Working on mathematics for oneself can, with appropriate reflection, enhance the possibility of noticing opportunities when working with others. For example, becoming aware of personal shifts from particular to general and from general to particular can sensitise us to the possibilities of provoking or noting similar shifts for students.

Ways of working

The group as a whole seemed content with mathematical generalisation in a global sense, but did not get down to sorting out categories of content which might perhaps require different forms of generalisation: rules, concepts, problem-solving involve different goals and or processes. Similarly, problem solving was used as if everyone knew and agreed what it meant, whereas this is problematic, especially within the context of generalisation.

Strangers thrown together in problem-solving mode contrasts the ease with which acquaintances can get started.

In the first task (unit fractions) we all understood the problem and worked on it collectively. In the second (Babylonian units) there was no common understanding of the problem and it was difficult to work collectively, to exchange productive ideas.

> The working group renewed my personal confidence in my mathematics ability, by being able to do some mathematics, supported by compatible and convivial people.
>
> I am frustrated (locked up energy) at trying to classify awareness, particularly as it seems to be intellectual, ignoring bodily, social, and community aspects of awareness. We are playing *male* games of 'my awareness can top your awareness'.

I do not think that we really listened to each other, but rather, prepared what we were going to say next.

As leaders we can only work with what people offer. Depth of insight depends on individual and group focus, only so much of which can be sustained by the leaders.

We could have worked for three days on the unit fractions, allowing ourselves to experience our *selves*, individually and in group, communally and socially. We shy away from the *becoming, the beholding of our awareness of our awareness*

LEADER REFLECTIONS

For JM this was the first attempt to use historical materials as the basis for a working group. There certainly were difficulties! Whereas a mathematical problem can usually be stated succinctly so that individual and group work can begin immediately, historical material has to be read, construed mathematically as well as historically, and perhaps set in context. It would be desirable to have the original sources, and to have time for people to read and contemplate them, but this would be time consuming and indeed often difficult to arrange.

As expected, there was more material than could be worked on during the time, but the notes offer participants an opportunity to extend the work they did at the meeting without having to go to other sources in the first instance.

No explicit attention was given to getting participants to get to know each other except as they worked together on tasks. Some find this a good way to meet, others wish for more gentle and explicit introductions.

In the general introduction it was suggested that each person would have brought what they wanted to work on, and at the time this seemed perhaps a little odd, as participants came to work on the working-group topic. But the comments show the considerable range of issues and sensitivities which participants bring: each person views the event through their own concerns and personal stressings, so that in very real sense, each person is working on their own issues, however related to the communal theme.

REFERENCES

Ballard, R. P. (1928) (1947 edition). *Teaching the Essentials of Arithmetic*. London: University of London Press.

Brown, S. and Walter, M. (1983). *The Art of problem Posing*. Philadelphia: Franklin Press.

Carr, G. (1886) (reprinted 1970). *Formulas and Theorems in Pure Mathematics*. New York: Chelsea.

Dyrslag, Z. (1978). O Poziomach Rozumenia Pojec Matematycznych, *Zeszyty Naukowe WSP w Opulu, Studia i Monografie*, 32, (list translated by Sierpinska, private communication).

Flegg , G. Hay, C. and Moss, B. (1985). *Nicolas Chuquet Renaissance Mathematician: a study with extensive translation of Chuquet's mathematical manuscript completed in 1481*. Dordrecht: Reidel.

Gerofsky, S. (1996). A Linguistic and Narrative View of Word Problems in Mathematics Education, *For the learning of mathematics*, 16 (2) p. 36-45.

Gerofsky, S. (1997 in press). The History of Mathematical Word problems and Their Pedagogic

Purpose, *Proceedings of History and Philosophy of Science Teaching Conference: Toward Sceientific Literacy*. Calgary AB: University of Calgary.

Gillings, R. (1972) (reprinted 1984). *Mathematics in the Time of the Pharoahs*. New York: Dover.

Heath, L. (1885) (reprinted 1964). *Diophantus of Alexandria: a study in the histroy of Greek algebra*. New York: Dover.

Herstein, I. (1964). Topics in Algebra. London: Blaisdell.

Hoyrup, J. (1990). Subscientific Mathematics: Observations on a premodern phenomenon. *History of Science* 28(1), 63-87.

Jones, S. (1912, 1927). *Mathematical Wrinkles: a handbook for teachers and private learners*. Nashville, TE: Samuel Jones Publisher.

Jones, S. (1912, 1927) *Mathematical Wrinkles: a handbook for teachers and private learners*. Nashville, TE: Samuel Jones Publisher.

Lakatos, I. (1976). *Proofs and Refutation*. In Worral and Zahar (eds.). Cambridge: Cambridge University Press.

Li, Y. and Shiràn, D. (1987). Chinese Mathematics: a concise history. In Crossley, J. and Lun, A. (Trans.), Oxford: Clarendon Press.

Loomis, F. (1974). *Calculus*. London: Addison Wesley.

Mason, J.H. (1997). Awareness and Expression of Generality in Teaching Mathematics, Draft Notes containing errors, infelicities etc.; an electronic copy can be obtained from j.h.mason@open.ac.uk.

Michener, E. (1978). Understanding Understanding Mathematics, *Cognitive Science*, 2 p. 361-383.

Mikami, Y. (1913) (reprinted 1974). *The Development of mathematics in China and Japan*. New York: Chelsea.

Polya, G. (1962). *Mathematical Discovery: on understanding, learning, and teaching problem solving*. New York: Wiley.

Reed, S. K. (1989). Constraints On The Abstraction Of Solutions. *Journal Of Educational Psychology*, 81(4), 532-540.

Swetz, F. (1987). *Capilatlism and Arithmetic: the new math of the 15th century*. LaSalle: Open Court.

van der Waerden, B. (1983). *Geometry and Algebra in Ancient Civilizations*. Berlin: Springer-Verlag.

Working Group B

COMMUNICATING MATHEMATICS

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INTRODUCTION

Writing this report poses an immediate dilemma for us: how to communicate through written text alone the purposeful exploration of a wealth of other means of communication? How to do justice to the intense co-emergence and co-working of ideas that took place? Perhaps we, Doug and Susan, need first to examine our interpretations of the purpose of a Working Group. For us, a CMESG Working Group is a starting point, a pause in the hurly-burly of normal life, an opportunity that can occasion deep thought, personal reflection and most importantly, considered, yet uninhibited, sharing of ideas. We had no aim to produce some "artefact-as-outcome." [1] The outcome for all participants was distinct and yet the same: personal interpretations of a shared fragment of history; a questioning anew of individual thinking through communal provocation. Our intention was to explore the intersection of representation, communication and mathematics— while bearing in mind that as a group we would have no common and clear definition of any of these terms! We did not see our role as chairing a discussion that would produce its own dynamic and go where the group would take it. We saw our role as guides through a structure that we hoped would challenge the thinking of the group participants and was based in Pirie's notion of "liberating constraints and constraining liberation." We aimed to blow apart the jigsaw puzzle of communication and focus on an examination of the individual pieces. So, faced with the tyranny of sequential text, we will do our best to give the reader a flavour[2] of the multiple and coincident workings of the group. It is clear to us that a "we did ... they said ...the conclusions were ..." report would impose a totally inappropriate strait jacket on the proceedings. Instead, much of the report will be quotations from group members, gathered at the meeting and through their subsequent reflections. Because we respect the trust and safety of the group environment as one in which to try on different ideas to see if they "fit," we will not attribute any of the comments to named individuals.[3] And because we have been asked to write the report we will not refrain from indulging in personal comments from time to time.[4] This, then, is our re-presentation of the living history of Working Group B. It will not make straightforward reading, but then the group did not work in straightforward ways.

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[1] Personal communication with B. Davis and T. E. Kieren.

To attempt to create the feeling[5] of the environment in which the participants initially found themselves, we will start at the very beginning....

A "RE-PRESENTATION" OF THE EXPERIENCE

Susan's Introduction:

To give you some idea of whether you wish to stay here or move at this point, let me give you some rough idea of what we have in mind for the three sessions. The working group title is "Communicating Mathematics" but we would like to get away from the over-worked, catch-all phrase of "mathematics is a language" and concentrate on the role of representations— whatever they might be— in the communication of mathematics. Our aim is to alert you to the notion of "problematising mathematical representation and communication in the classroom" and from there to provoke your thinking, possibly to even address the questions "what is 'mathematics' and what is a 'representation of a mathematical idea'?"

Some of the other "big" questions might be: In what ways can mathematics be communicated? When are particular modes of communication especially powerful? Do certain representations of mathematics have the power to conceal meaning as well as communicate mathematical understanding? Always in the background must be: What assumptions underpin any attempt to communicate?

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[2] S: Aha, now there's a means of communication that we did not consider! What is the flavour of mathematics? Sharp and cold like lemon sorbet, or warm and buttery like Werthers?
D: Certainly beats tofu! And what is the complementary scent of mathematics?

[3] ... but yes, participant, you really did say that! These were your very *words* - though of course your facial expressions, gestures, and tone of voice cannot be conveyed here.

[4] S: Doug and I had never worked or even really thoughtfully engaged with each other before we were thrown together by the CMESG committee in their wisdom, but out of the challenge to lead a group came a continuing working together - via e-mail, during breakfast, on holiday, across a hospital bed, around many cups of coffee, spanning two continents .... and, most importantly, we hope, over time.

[5] S: That reminds me of the possibility of communication through touch. If I handled mathematics what would it feel like? Oliver Sacks offered me an interesting metaphorical insight into linear equations through a tale of seeing and feeling (Pirie & Martin, 1997). But I digress.
D: One of the privileges of being a leader! Ihde (1979) explores embodiment relations in technology such as "(human-machine)-->world". Suppose we metaphorically 'replaced' the "machine" with "mathematics"; then what feeling?

To get us started Doug and I will each present a classroom incident and pose some of the questions that they raise relating to communication of mathematical meaning. We hope that you too will have anecdotes of critical classroom episodes of communication that you will share as we seek to define and characterise the variety of ways we have to communicate mathematics.

The incident I am offering concerns two students, Kevin and Dave, who were working together, making tables of values and plotting graphs. One of the equations given on the worksheet was $y = 2x+3$.

Kevin You fill it. [Referring to their table]
x - one, x squared - two, x squared plus three - five.
x - two, x squared - four, x squared plus three - seven.
x - three, x squared - six, x squared plus three - nine.
x - four...Wait, we need some minuses...er...do minus three.

Dave Easy-peasy. Minus three, x squared - minus six, minus three.
- minus two, x squared - minus two,...no...minus four, - minus one.
- minus one, x squared - minus two, - plus one.

At this point the teacher passed by and hearing "x squared", stopped.
Teacher Hang on. Go back a bit. You said "if x is three then x squared is six?"
K Yeah.
T OK. Hang on a minute...
He then proceeded to draw a 3x3 square, divide it up into unit squares, and count them.
T So it's nine, isn't it?
K What?
The teacher then repeated the explanation and demonstration with a 4x4 and a 5x5 square.
K Oh, yeah. [doubtfully]
T Can you carry on now?
K Yeah.
The teacher moved away.
K What's he on about? I know all that stuff about areas and counting things.
D Yeah, what we've got here's lines.
K Just ignore him. Your turn. You do x squared minus three [writing 2x-3]

The obvious modes of communication here are spoken language, symbolic representation and drawing. Kevin and Dave appear to be co-communicating mathematics with meaning, namely they are communicating verbally about the process of evaluating a symbolic equation. What 'assumed to be shared' knowledge is involved?

The communication between Kevin and Dave and the teacher is less straightforward. There is evident misuse of the conventional language to be accorded to a particular symbolic expression. Does this resonate, for the teacher, with a common student error?

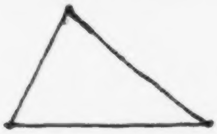
The teacher's talk and drawing resonates with some existing understanding that Kevin and Dave have. The drawing communicates meaning associated with the calculation of area, but not the intended verbal association of "square" with "squared". Why?

There is evidence that K and D have created links between graphical and symbolic representations, so we might say that the symbolic representation has communicated mathematical meaning, namely that $y = ax + b$ can be graphically represented by a straight line. What assumed shared knowledge is there in this interaction? Assumed and shared by whom? What are the effects on the understandings of K and D?[6]

Over to you, Doug.

Doug's Introduction:

My scenario has to do with teachers, but teachers as students. I have a middle school pre-service mathematics methods class and this incident occurred a few, 'extremely creative' classes into the geometry topic that we'd been doing. I drew a, so-called routine, symbol on the board, that is, a triangle, and actually I'd say that it turned out to be an acute angle scalene triangle but I realised when trying to draw, that it's not necessarily that[7] routine. Nevertheless here it is, on the board, something like so:



And I asked the clever question, "What have I drawn on the board?"

The intent of this open question was to solicit a range of responses and really open up a discussion of communication through pictorial representation. The teachers having gone through this period of classes with me, I anticipated that what I would get would be a variety of responses on this. Well, what I got in fact, was a fair bit of silence, looking around, anywhere but the front, and some of them thinking "there is possibly a trick question here." Eventually the response from somebody in the room was: "a triangle?"

I was trying to put them in a context where they were the teacher and they had grade six children in their class and they're going to be interpreting all kinds of possibilities and I hoped they would really get into "the experience."[8] Well, it didn't work like that! After the hesitant response-question of "a triangle?" I stopped asking questions. I'm not sure whether today, I would do it differently, but I decided then that I was not going to ask any more questions because I began to realise that they would be rather leading questions, that I'd have to start trying to get more specific in terms of what I was looking for, so I stopped, and I began a discussion— which I'm not going to get into here.[9]

Instead let me just pose some questions for you as part of your deliberations:

In a middle school class, what might the figure on the board represent? In a pre-service teacher methods mathematics class, what might the figure on the board represent? What's the potential for confusion?

How is it possible to determine "appropriate" representations or representation of representations?

Let me also pose a few questions about the question that I asked: What assumptions were imbedded in it? What strengths (if it had any) did it have? What weaknesses did it have? What might have been asked instead?

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[6] For further discussion on this topic, see Pirie (accepted).

[7] A conventional attempt to convey tone of voice in written reporting. But how many other subtle inferences have already been lost to the reader?

[8] See foot note 5.

[9] See Franks (1995) for more discussion.

S: .... and I would add to that: Is this (pointing to the board) actually a "representation of mathematics" or is this "mathematics"? What is representation in there?

S: With the theme of communication and representation, what we would like to do now is brainstorm as many different ways of "representing" as possible. We've clearly come up with some of them just in the two incidents we've shared. What we'd really like now is for you to think of all the possible ways that communication could happen. Communication of mathematics.

D: I'll write them on the board.

M: Are we putting down ways of communicating mathematics? [S: (nodding) Hmm] ways mathematics can be represented? [S: (nodding) Hmm] external representations? [S: (nodding) Hmm] internal representations? [S: (nodding) Hmm] all of them? [S: (nodding) Hmm]

M: Yes?

S: Yes!

M: Yes! [laughter] Silence.

Representational forms - Brainstorming - Day 1

- talk
- action
- diagrams
- picture
- gesture
- facial expression
- text
- symbols
- silence
- tables
- cheers
- sounds
- graphs
- video
- cartoons
- story
- visualizations
- journal writing
- singing
- pile of buttons
- computer screen image
- numbers
- chanting
- objects
- computer programs
- mental models
- transformations
- poetry
- intonation
- test scores
- physical models
- body language
- movement
- mental images
- mental operations
- talk (math language)
- talk (informal language)
- algebraic express^{ns}
- matrices
- story questions
- proof
- algorithmic procedures
- thoughts
- riddles
- metaphors
- analogies
- real life appl^{ns}
- simulations
- curriculum guides
- concepts

S: OK, let's hold it there for a minute then, because what we wanted to do now was have you pair up and have you focus on some of these in your pairs, and what we'd like you to do with the representations that you're going to get , is discuss what can be communicated through your particular representation and what can't be communicated. So, what's your particular representation good for, and what isn't it good for? Then we'll come together and at that point then we'll start unpicking these and talking about what we mean by representation and communication.

The Leaders

The allocation of forms of communication was totally random, because we wanted them to be jumbled and possibly incompatible. So we just numbered the groups 1 to 8 and then continuously numbered the words 1 to 8 until they were all allocated.

D: It is interesting that at least two of them have a strong urge to try to group the forms—not really sure what criteria though.

S: No, it is just a need to tie together that we are going to have to gently resist, since it is exactly counter to our intentions!

D: Do you notice that we use 'representation' and 'communication' almost interchangeably at times?

S: Yes, but I think that just points up the subtlety between the two in this circumstance.

D: 'Concept' and a 'pile of buttons': a bit out of left field...

S: ...and most of the groups were hoping that they would not get landed with the buttons!

Groups - Working Separately

List of the "forms" that each group had:

Group 1: talk, silence, visualizations, computer programmes, movement, proof, curriculum guides.
Group 2: action, tables, journal writing, mental models, mental images, algorithmic procedures, concepts.
Group 3: diagrams, cheers, singing, transformations, mental operations, thoughts.
Group 4: picture, sounds, pile of buttons, poetry, talk (mathematical language), riddles.
Group 5: gesture, graphs, computer screen image, intonation, talk (informal language), metaphors.
Group 6: facial expressions, video, numbers, test scores, algebraic expressions, analogies.
Group 7: text, cartoons, chanting, physical models, matrices, real life applications.
Group 8: symbols, story, objects, body language, story questions, simulations.

When we came back together again, each group did a verbal presentation using flip chart sheets (at the instigation of the leaders) and talk. The following snippets are to allow you access to some of the comments that arose during this activity and proved to be thought provoking—for us at least! (W=female, M=male)

> W: An issue we had was the meaning of the term of "representing something that somebody expresses" versus the "interpretation of something that is being represented by somebody else." They are two different ways of using communication. Mathematical symbols can represent mathematical objects, and operations and relationships and so on. They don't necessarily communicate an understanding.
>
> W: We found this was really embryonic thinking around these [representational forms]. We were noting that we weren't using the same sorts of words to talk about the different forms.
>
> W: So we got sort of confused when we got "objects" [on our list] because we realised that when we talked about symbols, the first thing we said was "mathematical objects," so we said well, objects could represent mathematical entities, and we were talking here about physical objects. The trouble is we both did a lot of work with Logo a while ago [engendering] a very interesting discussion for us around "objects."
>
> W: [Story questions] It's not just with the questions or the context but having kids write stories about mathematical ideas and their perceptions of the mathematical ideas and so becomes a way of communicating their meaning.

M: You pointed with your body language to feelings.

M: I'm convinced that you can't dissociate them, mathematics, at least at the personal level, from those feelings. [During a recent experience] just my feeling suggested to me that there was something about dealing with real numbers, especially with real numbers as computational entities, which was very distinctly different from dealing with things as abstract algebraic things, and that feeling wasn't dread but I went ..[gasps] you know I was surprised.

M: [indicating the flip chart] We ended up going to symbolic forms [pictures, algebraic expressions, etc.] here [because of] the fact that we couldn't agree on the words we were going to use! Facial expression doesn't exactly communicate content of mathematics, it tends to communicate the relationship between the mathematics, how you think about it, and you tend to reflect value by the expression on your face.

M: [video] To try and put [a mathematics lesson] on tape means that you would have to somehow imagine what your class would do in response. You can transmit information using video, but you can't really have a co-construction of mathematical knowledge.

M: I can't think of a single thing in mathematics that numbers can't communicate.

M: I don't know if you can say that algebraic expression does or doesn't communicate interpretation if you simply take it out of context.

M: The opposition [to mental operations] we saw here was that these were more for self-communication and I'm not usually ever properly communicating a thought directly to somebody else.

M: Diagrams are one means of communicating to somebody else, and one thing we felt they were doing particularly well was work on relationships that helped you to show relationships and one thing we felt they were doing particularly badly was things which had to be very strictly linear, like a linear formed chain of argument, like a proof.

M: [Cheers] was more like the intonation [which] by itself couldn't do very much. It's kind of an accessory, an overlay which can express the nature of math, beauty, motivation, relaxation, and the content is accessory. I mean the fact that I cheer doesn't convey any content. It gives a flavour to that content.

W: There's nothing a mental model can't represent to people. We actually decided somewhere along the line in our discussion that in many ways the only thing we put up here [on chart paper] that's a representation is really internal because it was physical things, tables, charts, to us in the way we interpret things that mathematics isn't in them, it's in the way we interpret them. For all mathematics representations they're really up here [pointing to her head], in the way we interpret the things around us. It's not inherently in that pile of buttons. Sorry to disappoint you guys.

W: Finally concepts. We just decided concepts themselves are not representations of anything therefore we're throwing that one off the list [laughter].

W: [The buttons]; that turned out to be our best category, for us anyhow. So, in the pile of buttons we could see just about everything. They represented the mathematics.[10]

W: All the mathematics.

W: We divided [text] into text as external objective authority which is monological and text which is of internal authority which is dialogical. So the first one, the external authority, was the textbook stuff where there's not really much interaction or response.

W: All the other ones we've written, they all came under the heading of an internal authority, and that includes informal writing, story writing, journal writing, and we thought with this internal authority comes a responsibility. If you've written the text you're the author of the text, you own the ideas in the text then you have the responsibility of justifying, convincing others of what you've written, so it becomes very interactive.

W: We got into talking about instrumental learning as opposed to relational learning. We think maybe the purpose of chanting is in enforcing things that maybe people have to know.

W: It's definitely of an instrumental nature.

M: The physical model can be used to represent most mathematical concepts. That's a big claim. Actually it's a statement of belief.

M: The human imagination comes into play here so of course, one can argue that underneath this is all the mental imaging. Actually I would not agree with that. I have become an enactivist supporter in this. I don't think it resides in the head.

W: We think, "Well, of course, real life applications." With our experience, and most mathematical concepts would probably be represented through some form of real life experience but then we thought, "Well, real life to whom?" Whose 'real life' are we talking about?

M: We thought [silence] was a critical lubricant or catalyst for mathematical ideas.

M: It seems to me that the silences in things can, in fact, at least invite one to contemplate the mathematics.

M: I think that this idea here [proof] is—and all the ideas, chanting [for example], we were just starting to get at what chanting might be.

M: So I think we've probably, in all of these, not gotten anywhere near where we could go.

The next Working Group activity was to actually try to communicate some mathematics using only the forms of representation that each group had been allocated. Before this task was introduced, we, the leaders, looked carefully at the compositions of the groups and the forms of communication that they had been exploring and selected mathematical topics that we thought would be challenging to think about when one was constrained to use only the restricted set of means of communication.

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[10] D: Looks like the previous group doesn't need to "apologize" after all!

Groups - Working Separately

List of topics given to the groups, who could work only with the representational forms assigned earlier (with some re-organization of groups):

Group 2: trigonometry -- cosine law

Group 3: division of fractions

Group 4: flips, slides, and turns

Group 5: addition of whole numbers

Group 6: calculus - differentiation

Group 7: area of a circle

Math Topic Presentations

Group 2 opened to us the diaries of two high school students writing to an ailing friend about learning the cosine law in math class that day. Group 3 dramatically sang and vigorously sketched, under enthusiastic exhortation, to reveal comparison as fundamental to division of fractions. Group 4 challenged us to flip, turn and slide poetic images, shiny transparent ducklings and a brightly coloured row of buttons. Group 5 asked us to enter the domain of computer images, graphs and video game skill development to gain an understanding of the addition of whole numbers. Group 6 moved silently about the room and into the limiting process and the notation of calculus. And en'chant'ing Group 7 textualized group activity procedures for understanding circle area as a transformation of the parallelogram (and struggled with limited voice and no diagrams.)

The Leaders

D: One thing we had not allowed for was a group that had someone who was not familiar with the topic very strongly requesting a change. Lucky we had one to spare!

S: But members of another group were in the same situation and set about teaching the topic to their third member.

D: Well, fate really defeated us in respect of one participant!

S: You don't think he killed off his two previous partners in order to get the topic of division of fractions? In any event, he claimed that the experience had allowed him to see rational numbers in a new light with respect to ratios.

For the final part of the working group's time we re-convened as a whole group and tried to use the preceding days' working, thinking and listening to grapple with the bigger questions that were posed at the start of the sessions. Again we will simply give you, the reader, some of the highlights to enable you, too, to engage with the problematic areas that were our arenas for discussion.

W: I thought to be able to represent something in a different form we needed to know our area quite well. It caused us to think about the concepts differently. I think that just sort of ties into what it means to understand. I found that very powerful, you know. Just thinking about it that way, was something I hadn't thought about.

M: So you really had to know the concepts involved in order to be able to go beyond giving back the standard routines.

W: There was that communication where we were very much talking about our procedure, and then there were different types of communication happening here when we were looking at context. I don't know what it means, communicating mathematics, or mathematical communication or communication about that math.

W: I think maybe that there are fundamental differences and probably what as teachers we're trying to do is communicate mathematics. What was coming clearly out of those two letters [the group presenting journals as a way of communicating] was kids communicating about mathematics. And maybe we don't do that, in the classroom, we don't address the communicating about mathematics.

M: If that letter was there from a teacher it might be quite hard from the sterile reading of a sterile letter to communicate sterile mathematics [laughter]. If it's from a friend and there is a relationship then the text might work.

M: The constraints provided, I call it the pretext, for, at least for me, thinking about the mathematics differently than I had before.

M: Maybe we wouldn't have made up a song if we didn't have these constraints. We felt we needed to say something with it [their diagram]. He said if the main word I need is "compare", how do we bring "compare" in? Well, we only had the song, but we might have done even better not in a song but in some other way. So I'm not sure the constraints helped a lot.

W: Or was it having to do the song that made him focus on a word?

M: I was just thinking about that. You hit it right on the head. I was thinking you know this exercise was more valuable than I thought. [laughter]

M: When their group presented the limiting process I felt they weren't given enough means and they could have done much better if they hadn't been so constrained.

M: We know that all of us have been in mathematics classes where we have not been offered decent ways to think about anything, but we somehow said "well it's our job somehow to figure it out". We were secure, we already had enough knowledge about what was going on to say "well I think I can go in this way" but if you really don't perceive yourself as that, and you're insecure you'll say, "can't do anything here, I don't get it, the teacher's being unfair," and you start maybe building up other defence mechanisms. You take yourself out of the mathematics and put yourself into some other game.

W: They actually do more than that, they don't say "I don't get it", they say "I can't get it".

M: That makes it acceptable now. I'm not responsible for doing anything.

W: Just as we've gone through an exercise here where we've been almost forced to think of other modes of communication and it's opened up for me a lot of ways I really hadn't considered using as a way of expressing mathematics, I think we need to do that with our students, too. When we got "text." I felt like this is great, we've got a super way of communicating, those poor people out there who didn't get text. But we realised that we didn't have pictures, we couldn't really draw. So it was very limiting but it was a very good exercise to see that powerlessness, almost, of text. It has power but in our situation, it wasn't that effective.

W: Text isn't an immediate form where you can see how the reactions are coming and then change it, the subtle changes that you need to make when communicating.

M: Does representation become the mathematics, for many people? There is a question of meaning inherent in the symbols versus meaning external, or in other words, these things don't have any meaning until we attach some meaning to them and obviously then it's a question of who we are and what we understand of the mathematics. There is the question of *the* mathematics versus *representing* the mathematics. And that leaves some questions that Susan had earlier, concealing versus revealing. Given that we were doing things that we would never normally do, we've in fact revealed more than we would often, or otherwise, reveal.

M: In the instantaneousness of technology something is lost.

M: Affect versus content. Some forms of communication, representation, are excellent for conveying affect but they don't convey content.

M: There is the notion of show versus communicate.

M: But maybe we just want to find other words that can communicate to generate not to represent. Maybe we say or write things to provide presence for mathematics.

W: There is the situation as with Doug's triangle where the sketch is intended as some aide-memoire and the students take it literally.

M: ... as if it were text, but this isn't meant as a representation.

W: I think that gets to the heart of it, because any representation must have things you don't want the kids to take with them.

W: True of any form of representation and we don't stop and think what it is I don't want you to take out of this.

M: True but any representation is also always missing part of what you would like it to say.

M: Context is critical in all this. This [Doug's] figure, it could just be a representation of any old triangle. As a triangle, it stands for all possible triangles that one could imagine. Or it could be, in fact, given the context, a representation of a particular triangle with those particular measurements, with those particular angles and lengths of sides. Or it could be a representation of a particular of class of triangles, scalene triangles, or acute angle scalene triangles.

W: [What are] the assumptions, we have? We made the assumption he doesn't want us to say just "triangle."

For other participants the image on the board carried a variety of meanings:

W: ...the triangle was used to memorise how to work with interest equals PRT ... Maslow's hierarchy. ... my Ed. Admin. class ...

Doug: Suppose I'd drawn an obtuse angle triangle, would you have not found those things?

W: What I'm interested in is exploring and looking at how I've grown into what I've learned.

By the end of the sessions there was still, however, this lone voice:

W: I wanted to do a concept map, of all of these ideas on the wall, and I wanted us to cluster them, I wanted us to think of different ways that we could group them. That would be a great way to report to other people who have not been a part of these meetings.

W: I don't know that it would be a good way of representing what we did because it isn't actually what we've been doing for the past three days.

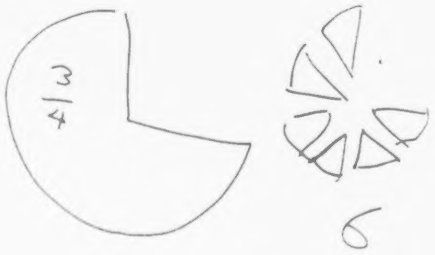
M: Actually I tend to agree because I think that there's things that came out of here which totally surprise me, you know, my own response, I didn't think I would think about these things, putting myself in the mind of a student, trying to see the constraint problems from their perspective, and here we put ourselves in that position, and that to me seemed to be a more important issue than how you can connect these things. You know there's so many different ways in which you can break it up, communication, I was sceptical listening to "silence" and all this stuff. I hung in because I thought, well, you know, I've got confidence that people at CMESG are going to come up with something, and I think it was well worth the effort because what I saw was that people were genuinely coming to grips with a difficult question, and making things that we assume, very problematic.

M: That's always a very useful thing, the problematics of the situation that we came to grips with.

M: If anything should come out of here it's our understanding what the problematics are.

We closed our report to the full meeting of the CMESG with a repetition of one group's presentation. That group had as their allowable forms of communication: diagrams, cheers, singing, transformations, mental operations and thoughts. We will close here with all that remains when we work with the further constraints of the mediums available in this report:

M: [singing:] We have some pizza, we have some pizza, we have three-fourths!
W: [cheering:] Come on Tom, come on Tom!
M: We wanna make pieces, we wanna make pieces!
W: Come on, Tom!
M: They should be of size one-eighth.
M: How many pieces? How many pieces?
W: Come on, Tom! Tommy! [shouts ha-di-ya!]
M: We must compare! We must compare!
W: Come on, come on Tom! [ha-di-ya sonneeee! ha-di-ya sonnee!]
M: We have our portions, we have our portions.
W: Come on, Tom!
M: And it is six. It is six.
W: That's it! We have made it. We have made it! Whew!



REFERENCES

Franks, D. R. (1995). Reflections of a developing constructivist mathematics teacher educator. In D. Franks, H. Langford, and W. McEachern (Eds.). *Constructivism: Its significance for education (symposium proceedings and related papers)* (pp. 73-89). North Bay: Nipissing University.

Ihde, D. (1979). *Technics and praxis*. Dordrecht: Reidel.

Pirie, S. E. B. (accepted). Crossing the gulf between thought and symbol-language as (slippery) stepping stones. In M. B. Bussi, A. Sierpinska, and H. Steinbring (Eds.). *Language and communication in the mathematics classroom*. Reston, VA: National Council of Teachers of Mathematics.

Pirie, S. E. B., and Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics, 34*, 159-181.

Working Group C

THE CRISIS IN CURRICULUM

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INTRODUCTION

At the 1996 meeting of CMESG a panel considered the question "Who drives the curriculum?" While it was clear that this question is a complex one, there was some agreement that CMESG/GCEDM could (and perhaps should) play a part in future curriculum change in Canada. In light of this, Malgorzata Dubiel and David Reid were asked to organize this working group on the theme "The Crisis in School Mathematics Content."

David and Malgorzata had big hopes (and worries) that we would solve the crisis during the first day, and be left with nothing to do for the remaining time. However, as often happens at CMESG/GCEDM working groups, the strong feelings and opinions the group members brought with them took over, and the result has only a slight resemblance to what was anticipated by the "leaders". We haven't reached the point, suggested in the conference announcement, of initiating a work towards the content for a curriculum appropriate to the foreseeable future. We concentrated, instead, on analyzing thoroughly complex aspects and components of the present crisis. We formulated many questions which arose in our discussions, but time did not allow us to provide many answers. We can only hope that our work will be continued at another CMESG meeting, and that our report will provide a background for a group who will take over.

In our three days of meetings we heard reports of curriculum issues from across Canada and around the world, considered the many issues which form a part of the crisis of content, and discussed ways in

76

which we could effect changes which might alleviate the "crisis in content." This report does not attempt to reproduce everything that was said during the three days in Thunder Bay. Instead we have tried to present the ideas that were discussed. We have used participants' comments when needed to illustrate and explain these ideas. The report is not written in chronological order, but instead attempts to re-present the discussion around several themes. These include the nature of "content," the curriculum as intended, taught and tested, the need for and nature of "standards," the impact of universities on curriculum, preservice education of teachers, support for teachers in the field, and the impact of technology. During our meetings, we condensed parts of our discussions into short phrases to serve as seeds for further discussion. The section headings in this report are drawn (for the most part) from these seeds.

A. WHO ARE "WE"?

When we came to consider what we might do in order to address (if not solve) the "crisis in content," it became important to consider who we were, who we claimed to represent, and what our personal goals might be. One way of describing us would be to list our occupations. We include school teachers of mathematics and other subjects, school board consultants, university professors teaching mathematics and other subjects such as economics and biology, professors teaching mathematics methods courses to prospective teachers, and others. Another way of describing us would be to consider where we were. We had traveled to Thunder Bay from across Canada, and from France, Israel, Italy, Kuwait, and Singapore in order to participate in the annual meeting of the Canadian Mathematics Education Study Group. That indicates both our interest in mathematics education, and our interest in sharing ideas with colleagues from across Canada. It also indicates that most of us subscribe to views of mathematics and education that are different from those of many people, including government officials involved in curriculum design and describing us is as individuals, whose work in mathematics education and participation in the working group is motivated by a whole constellation of personal interests.

We may have different opinions about many aspects of math curriculum in Canada, but we all share the determination to work towards improving it, making it more meaningful and relevant to students. The fact that we all traveled to Thunder Bay and became involved in this working group indicates that we agree that there is a crisis in high school curriculum, or at least are open to this possibility. What is not necessarily as clear is the nature of this crisis, and the remedy required.

B. CURRICULUM CRISIS I: IS THE CRISIS 'IN' CONTENT? AND WILL CHANGING THE CONTENT CURE IT?

Anna raised the question: "Why do we keep making changes in school curriculum?" She pointed out that we need to know the laws of the system we want to change before deciding what to change and whether it is possible to change it at all. We could look at an educational system of a country or province or a culture as a dynamical system with a number of fixed points. We have to know where these fixed points are and why some of them are strong 'attractors'. The reason why the changes that we introduce into the system (varying the content, introducing small group work in the classroom, other things like that) do not ultimately change anything in the system is because these changes do not destabilize the system: everything converges to a fatal fixed point! She mentioned some possible candidates for laws of the system: school mathematics is essentially different from research mathematics, teachers can be trained to be reflective practitioners but they will not be researchers in mathematics education (they want to have 'life after school'), any change whose beneficial effects can only be seen in the long run is not likely to be accepted by teachers, students' attitude towards knowing and value of education reflects the society's culture, etc. Of course, whether these are actually "laws" and what their domain of validity might be is a problem for research.

Many curriculum changes which were introduced over the last 30 (50?) years, seem, in retrospect, to be exercises which only brought confusion and frustration to students and teachers, and contributed more to creating a crisis than curing it.

Questions of whether there is a crisis, and if so, what is/are the reasons for it, reflect both a need to understand the complexity of the present situation, and a frustration with outcomes of all the previous curriculum changes. We never managed to formulate a satisfactory answer to them (if there is one), but instead focused on discussing two of its components: content and delivery.

B1. WHAT IS CONTENT?

During our discussion we began to distinguish between "the list" (roughly what is called the syllabus in the UK and Australia) and a wider sense of curriculum content. This distinction allowed the question to be raised (by Doug) whether our object was to change the content on the list (which we saw as largely skills and facts), or to integrate processes into the list so that it would reflect our larger vision of curriculum content.

In order to distinguish between the limited content (exemplified by "the list") and a more process oriented vision of content, Gary proposed the word "protent." As Gary puts it:

Protent = 'Pro'cess + Con'tent' + 'P'urpose/In'tent' + 'P'icture + 'P'erformance

Process: problem-solving, creative thinking, logical reasoning, use of technology, connections

+

Content: learning outcomes expressed as actions on objects (actions = explore, model, formulate, manipulate, transform, infer, conclude, communicate; objects = mathematical objects found in the various strands of mathematics)

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Purpose/intent: need/importance/relevance/application of concepts/skills/habits of mind self-evident through interaction with context/problem situation/rich learning task

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Picture: descriptions of sample context in action in a classroom, including actions and interactions of students and teachers with context

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Performance: sample learning/assessment tasks, including sample student responses and descriptions of levels of performance

One of the reasons we do have a curriculum crisis is the narrow understanding of content as a shopping list of skills to be mastered by children, which is held within many constituencies, notably politicians and some vocal parent groups, and pressures put on schools to reinforce these. As well, for some teachers who are not confident enough about their mathematics knowledge and understanding, such an approach may be the only possible one. Not to mention the fact that it is much easier to test "content" rather than "protent"; to test basic skills rather than problem solving, thinking and like.

C. CURRICLUM CRISIS II: IS THE CRISIS IN DELIVERY?

At many points in our discussion we came back to the difference between the curriculum as intended (roughly embodied by curriculum documents) and the curriculum as actually taught in classrooms. This issue was important in many ways. Clearly changing curriculum documents was meaningless if nothing changed in classrooms, because of teacher attitudes, lack of preparation, lack of resources or other reasons. Equally clearly is the fact that students' and parents' attitudes towards education constrain the effect that curriculum change can have. Perhaps paramount was the matter of testing, as we agreed that what is evaluated has a strong effect on what is taught, and how it is evaluated has a profound effect on how it is taught. A final issue related to delivery which we considered was the role of technology in changing what is taught and how.

In order to better see how delivery can be a factor we looked at several different models of curriculum (content and delivery) in Canada and around the world.

C1. INTENDED VS. ASSESSED VS. IMPLENENTED CURRICULUM

Lynda provided us with data from the recent assessment of grade 12 student achievement in Scarborough. Her data indicates a huge gap between intended curriculum and achieved curriculum, due to variations in implementation. About 50% of the curriculum was implemented, mostly at the skill level and an average of 17 out of 70 minutes of class time was actually used for instruction. About 30% of class time was taken up with tests and quizzes. This gap with the intended curriculum is not perceived by teachers, who reported that there is a lot of technology used and lots of active learning going on in their classes. Students, on the other hand, while being generally confident and positive, described math class as a very passive environment.

Lynda's description and documentation of the problems in Scarborough were very useful to our group, and we all agree with Bill H.'s observation that Scarborough is very brave to actually show that there are problems. We all know that there are problems and in most jurisdictions the problems are actually worse. Lynda's proposal of this sort of assessment as a way to drive change in other jurisdictions was an important step towards establishing ways in which we can change the curriculum as implemented.

An important implication of teachers' failure to implement parts of the curriculum, especially those related to higher level thinking, student activity, and technology use is that of student motivation. Traditional mathematics teaching methods are often dull, and in the absence of the sort of cultural valuing of education for its own sake seen in Singapore and other parts of Asia, this can leave students uninterested in mathematics.

C2. TEACHERS WITH PROBLEMS, PROBLEMS WITH TEACHERS

While Lynda showed us that there were serious gaps between intended curricula and what was delivered, others observed that the intended curriculum can be an impediment to the delivery of mathematical protent. Tom observed that in British Columbia there is actually a lot of overlap between what is written in the curriculum document and what is actually taught. This works if we want a top down model for our educational systems, but given that one of Canada's strengths is its well educated teaching force (compared to many other nations), it might be wiser to distribute control of curriculum more broadly. A top down approach makes it difficult for teachers to act creatively. What we need is room in the curriculum for teachers to do something interesting to themselves and their students. Emilia commented that she had had the opportunity to teach in a program that gave her control over part of the content, and it was a great experience for her and her students. The real shame, she said, was that her other students couldn't have the same sort of experience because of the constraint of the provincial curriculum. Bill H. proposed that whatever curriculum model we might settle on should include a mix of structure and freedom. Tom O'Shea mentioned the idea of a curriculum that was relatively specific for 4 days and had a 5th day where the content was optional(but mathematics). This gives both the flexibility to meet "standards" as well as provide teacher controls.

We cannot overlook the fact that teacher preparation has a big impact on what is happening in a classroom. Mark Twain's observation that "teaching is the fine art of imparting knowledge without possessing it" reflects many people's views of teaching, and, unfortunately, too often the reality. It is important to remember that, it is difficult to be creative if one does not have a sound knowledge of a subject and a lot of confidence. In Canada, in spite of the high level of education of teachers overall, there are a significant number of math teachers in the school system who do not have a strong mathematical background, and there is no adequate mechanism in place to help them.

Malgorzata related that one of the amazing aspects of the Australian curriculum change, and probably the main reason for its success (if only relatively short-lived), was the teachers' enthusiasm for introducing it, for professional development opportunities, for the opportunity to use their creativity—and resources from the federal government to support professional development within individual schools. This shows that exciting things can happen—but only if teachers are part of the change. But the opposite is also true: teachers become part of the change only if they embrace it.

C3. ROLE OF TECHNOLOGY

The role of technology was part of the original description of the business of the working group prepared by Malgorzata and David and it was a recurring theme although one about which we were not all in agreement. Opinions ranged from George's comment that technology should be downplayed in curriculum to the initiatives in the curricula proposed by the Western Consortium and the Atlantic Provinces Education Foundation which assume extensive technology use in all classrooms.

A key point was raised by Gerard in his description of curriculum change in France. He noted that calculators are everywhere and in some cases the calculators are ahead of the teachers. Students are doing their own mathematics with calculators which differs from the teacher's mathematics to such a degree that there is often no contact between the two. This has led to a debate on the impact of the calculators on concepts. Calculators have changed the concepts the students develop. Software like Derive increases this problem. For example, because calculators and computers terminate decimal expansion of numbers after a fixed number of decimal places, the concept of infinite decimal expansion, of infinity, and of limit, may be very different for a student and a traditionally educated teacher. This implication of technology use signals a danger, which is being felt in France, that teachers will be unaware of or unable to cope with students' technology based concepts. Moreover, we need to realize that overemphasizing technology has a potential to make mathematics itself disappear, if a tool becomes the content.

Another side of this danger is that technology is perceived by some (politicians, parents, even some teachers) as a panacea for all problems within the school system: belief that if schools have enough computers, internet connections etc., teaching and learning will instantly improve, and students will be better prepared for entering a workplace. While this all may indeed happen if we introduce the new tools wisely, we have to make sure that issues like problem solving, logical thinking, creativity and making connections—all which are high on most employers' wish lists—will not be neglected.

The issue of the role of technology seems largely unresolved. In some cases there is little acknowledgment in curriculum planning of the effects technology has already had on the world within and outside the classroom (as Gerard noted). In other cases the curriculum has a large role for technology, in spite of the lack of resources in schools (as in BC and NF).

C4. EXTERNAL STANDARDS

Lionel pointed out that a significant feature of education in Singapore is the clear external standard which students are expected to achieve. Bill Otto related this to the external standard provided by provincial examinations, which change the dynamic of teaching from the teacher having control of both the resources available to students and the evaluation of them, to the teacher and students working together against a common enemy. The lack of clear external standards was one aspect of contemporary schooling in Canada which George had criticized.

We spent a lot of time discussing different models with different approaches to standards or lack of them, without coming to a clear conclusion. The examples of Singapore, Sri Lanka, and the IB program certainly showed that external exams do provide a standard, however these examples do not translate easily into the Canadian situation. There are significant cultural differences between Canada and Singapore. While both countries want a well educated population, they differ in the their approach, with Canada having a broader program for all students. The IB program cannot be extended to the whole population as one of its features is that the teachers and students involved are volunteers. Moreover, its standards and requirements may not be appropriate for all students, and possibly too demanding for many teachers. Doug also pointed out that the internet makes administering a common exam in different times zones difficult. This problem could occur even in cases where one jurisdiction held it exams in the morning while another waited until afternoon.

An additional problem with external examination is the problem that the "protent" which we would like to see being a significant part of the curriculum is difficult to test in a large scale, externally administered examination. An area which might be worth more consideration is whether there are examples of external standards which do include protent.

D. FOR WHOM IS THE CURRICULUM?

A natural way to answer this question is—or should be—that it is for students, to prepare them to live in the world they enter when they graduate; that it should make learning meaningful and exciting, and motivate them.

Another way to read the question is in terms of the elements of society which can exert pressure on educational systems, and hence must be considered in designing curricula. Among these are politicians in ministries of education, politicians in ministries of finance, and business and industry in their role as potential employers of students.

A third way to read the question is in terms of the kinds of students curricula are designed for. This issue came up in a couple of contexts. One was the contrast between educational systems which are intended for an elite (as is currently the case in Singapore, and as was recently the case in France and parts of Canada), and an educational system for the masses (which is the current ideal in Canada). The other was the requirements of universities for particular content to be covered prior to university entrance.

D1. STUDENTS' NEEDS

When listening to various discussions about curriculum, one hears the voices of parents, teachers, educators, university professors and politicians but it often seems that those who are directly affected by the changes are somewhat neglected. Nobody asks the students' opinions—everybody seems to know what is best for them. When someone does ask (as Lynda did in Scarborough), it often turns out that those who claim to know what is best don't. In the Scarborough assessment, for example, the teachers thought the assessment was too hard for grade 3 students, but the kids liked it. So, at least in some cases, teachers don't know what students are capable of.

From Marilyn's perspective as a teacher, however, it is parents that underestimate the kids; they don't want teachers to be too hard. At the same time most parents want their children to go to university and so put them in university oriented mathematics programs, when in reality only about 20% of students actually go to university, and the others might be better served by a different sort of program. Parents and politicians often believe that a magic formula of "back to basic skills" is what students need. They speak of raising goals, but as Gary pointed out, their idea of raising goals is to go from one digit long division to multi digit long division. What students actually think about this may be reflected by Bill Otto's observation that most students enter primary school eager to learn and liking math. Somewhere, somehow, they loose this and the attitude becomes a big problem.

So, what do students need? Part of the answer was pointed out by Lionel: The big question is what math is needed to function in a society? And what should you teach to all?

Gerard and Torri described the difficulties being faced in France and Italy as they attempt to move from an elitist educational system to mass education. As Bill Higginson pointed out, Canada's commitment to educating everyone to a fairly high standard is one of our strengths, but also a challenge in terms of curriculum. Should we teach university preparation courses to everyone? Should we ensure that the mathematics required by certain professions is taught? If so which ones? Should we try to have students experience mathematics as mathematicians do (as Rafaella Borasi suggested in her plenary)? (While we did not come to definitive answers to any of these questions, having students experience mathematics as mathematicians do was generally supported.)

The commitment to educate all students results in the problem of what to do with high school students who have not achieved as much as their peers in mathematics. The BC curriculum's two streams are intended to be different in focus, not differentiated in terms of prior student achievement, but many teachers are concerned that the *applied* stream will absorb most of the low achieving students, to the detriment of the program as a whole. Lars informed us that Manitoba extended the Western consortium curriculum by adding a third stream for low achieving students, to address this problem. The question could be asked however, whether having a special stream for such students lives up to the ideal of educating everyone in mathematics that will have significance for them in their lives beyond school?

Mass education can also leave out students who achieve—or are capable of achieving—more than their peers. Some of these, not finding enough challenge in class, drop out or don't do work they find boring. Others, in fear of being labeled as "nerds" by their colleagues, trade in high grades for social acceptance. Others again may even loose faith in their abilities after getting in trouble with overworked teachers who are not able to recognize the quality of their work. This is one of the reasons for popularity of programs such as IB, which are often a refuge for these troubled over/under achievers.

D2. ROLE OF UNIVERSITIES

In a system in which only the elite complete 12 years of academic education, most are likely to attend university. In such a system university entrance requirements have a strong influence on curriculum, as has historically been the case in the UK, the USA, Canada, France, and Italy, and as continues to be the case in Singapore, Sri Lanka and China. Given Canada's commitment to mass education, however, the questions arose "What do universities expect from schools?" "What /should/ universities expect from schools?" and "What should schools offer universities?"

The issue of universities' influence on curriculum has recently been raised in BC as a result of UBC's and SFU's reactions to the new Western consortium curriculum. The requirement of the Ministries of Education in western Canada that there be no more that 20% overlap between grade 12 courses in the *Applied* stream and the *Principles of Mathematics* stream makes *Math 12 Applied* unsuitable as prerequisite for calculus. In British Columbia, most universities do not accept it as one of the required entry courses. As a result of this, most students would opt for the *Principles of Mathematics* stream even though only 20% go to university. This undermines the planners' target of 30% in the top stream. At the same time, lack of a third stream, suitable for weaker students, fuels the fears that the standards of the *Applied* stream will suffer. This has led Manitoba to add a third stream to the Western consortium's model.

E. WHAT CAN WE DO?

In the course of our discussions we came to some agreement on what "the crisis in content" meant, and what should be done about it. But this raised the question "Can *we* achieve it?," and a discussion of whom *we* should include followed. What should *we* become? Should one of our goals be for us to grow?

Should part of what we need to do be to keep in touch and involve more teachers and other educators?

E1. PRESERVICE EDUCATION

One area in which we have a great deal of influence is preservice education as many of us are directly involved in preservice education. Ann suggested that pre-service "protent" must change hand-in-hand with any curriculum change—or even before it. As Rafaella's video pointed out, one can't teach in a way one hasn't experienced learning. Pre-service teachers *must* realize and experience this "protent" and see that it IS the content. Many of us have taught pre-service math courses in which the students all sigh and say 'but this isn't math' about our "protent." We need to convince them it is.

E2. CURRICULUM TO CONSTRAIN TEACHERS OR EMPOWER TEACHERS?

Concerning curriculum change, Gary suggested that, in part, we can help by:

1. Sharing and discussing a vision of the math classroom with teachers, parents, politicians and business. Materials such as the four brochures developed at the 3rd meeting of the National Mathematics Education Institute (NMEI) are an example of this.

2. Discuss/lobby for a more enlightened way of writing curriculum policy and related provincial/board/school documents, so they are not read or interpreted or implemented as an endless sequence of bits of content.

3. Help disseminate and develop rich learning and assessment activities, compatible with the above. Examples can be found in the NMEI brochures, the proceedings of the 4th NMEI, and Peter Taylor's In Process textbook.

4. Lobby for more appropriate resources (e.g., new generation of textbooks)

5. Proceed with organized abandonment of many sacred bits of old curriculum and challenge old myths.

Several group members commented on projects (in addition to the work of NMEI) which are in keeping with these aims. Lynda's proposal of the kind of assessment done in Scarborough as a way to drive change in other jurisdictions was an important step towards establishing ways in which we can change the curriculum as implemented.

E3. RESEARCH IN CURRICULUM CHANGE

A final, and self-referential, step toward curriculum change is the work of CMESG and other professional associations to disseminate information, conduct research, and provide fora for discussion such as our Working Group. CMESG is uniquely positioned for this task: it is a national organization, its members include teachers, math educators, and mathematicians, and the main thing they all have in common is deep interest in the quality of mathematics education in Canada.

In future, we hope that model curricula, research findings on the organization of school mathematics, research on the social context of curriculum change, and additional materials will come out of the ongoing work of the members of CMESG on content and curriculum.

TOPIC SESSIONS

Topic Session 1

ABSTRACT ALGEBRA: A PROBLEMS-CENTRED AND HISTORICALLY FOCUSED APPROACH

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I want to describe a course in abstract algebra which I taught in an In-Service M.A. Programme for teachers of mathematics at York University. Students do not follow this course with another in abstract algebra, so I was fortunate in not having to worry whether I had covered this or that material for the next algebra course. This presented an opportunity and a challenge: What are some of the major ideas of abstract algebra that I would want to impart? What algebraic legacy would I want to leave the students with? Since the students were high-school teachers of mathematics, I wanted the course also to have at least broad relevance to their concerns as teachers.

All this suggested (to me, at least) that the history of mathematics should play an important role in the course. History points to the sources of abstract algebra, hence to some of its central ideas; it provides motivation; and it makes the subject come to life.

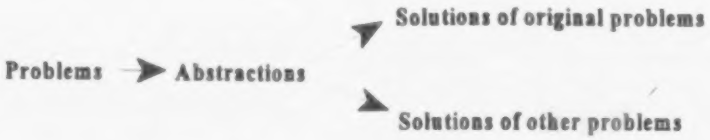
To set the context for the course, here is a history of abstract algebra—in 100 words or less.

Prior to the 19th century algebra meant (essentially) the study of polynomial equations. In the 20th century algebra became the study of abstract, axiomatic systems such as groups, rings, and fields. The transition from the so-called classical algebra of polynomial equations to the so-called modern algebra of axiom systems occurred in the 19th century. Modern algebra came into existence principally because mathematicians were unable to solve classical problems by classical (pre-19th century) means. They invented the concepts of group, ring, and field to help them solve such problems.

End of history; back to pedagogy.

The upshot of this mini-history of algebra is to help focus on the major theme of the course, namely showing how abstract algebra originated in, and sheds light on, the solution of "concrete" problems. It is a confirmation of Whitehead's paradoxical dictum that the utmost abstractions are the true weapons with which to control our thought of concrete fact.

Schematically, what I do in the course can be represented as follows:



The item "Solutions of other problems" is intended to convey an important idea, namely that the abstract concepts whose introduction was motivated by concrete problems often superseded in importance the original problems which inspired them. In particular, the emerging new concepts and results were employed in the solution of other problems, often unrelated to, and sometimes more important than, the original problems which gave them birth. I will call the solutions of such problems "payoffs." But now to the problems.

PROBLEM I: Why is (-1)(-1) = 1?

This problem is an instance of the issue of justification of the laws of arithmetic. It deals with relations between arithmetic and abstract algebra, and it gives rise to the concepts of ring, integral domain, ordered structure, and axiomatics.

The above problem became pressing for English mathematicians of the 19th century, who wanted to set algebra (to them this meant the laws of operation with numbers) on an equal footing with geometry by providing it with logical justification. The task was tackled by members of the Analytical Society at Cambridge, notably Peacock. Before proceeding with a modern treatment of the topic, I discuss with students Peacock's work, *Treatise of Algebra* (1830), which proved very influential. Its significance, certainly from a modern perspective, was to have symbols take on a life of their own, becoming objects of study in their own right rather than a language to represent relationships among numbers. Some have said that these developments signalled the birth of abstract algebra.

We next discuss a modern, Hilbert-like approach to the above topic. The idea is to define (characterize) the integers as an ordered integral domain in which the positive elements are well ordered, just as Hilbert (in 1900) characterized the reals axiomatically as a complete ordered field. Of course, in the process we must define the various algebraic concepts that enter into the above characterization of the integers. We can then readily prove such laws as (-a)(-b) = ab and a·0 = 0.

Payoffs:

The following issues arise from the above account:

(a) How can we establish (prove) a law such as (-1)(-1) = 1? This question leads to axioms. (We cannot prove everything.)

(b) What axioms should we set down to give a description of the integers? This question enables us to introduce the concepts of ring, integral domain, ordered ring, and well ordering (induction).

(c) How do we know when we have enough axioms? Here we introduce the idea of completeness of a set of axioms.

(d) What does it mean to characterize the integers? This sets the stage for the introduction of the notion of isomorphism.

(e) Could we have used fewer axioms to characterize the integers? (For example, a + b = b + a is not needed.) Here we come face to face with the concept of independence of a set of axioms.

(f) Are we at liberty to pick and choose axioms as we please? This question permits us to introduce the notion of consistency, and more broadly, the issue of freedom of choice in mathematics.

The innocent-looking problem (-1)(-1) = 1 can be a rich source of ideas!

PROBLEM II: What are the integer solutions of $x^2 + 2 = y^3$?

The problem deals with relations between number theory and abstract algebra, and it gives rise to the concepts of unique factorization domain and Euclidean domain—important examples of commutative rings.

The main idea in solving $x^2 + 2 = y^3$ is to factor the left side: we get $(x + \sqrt{2}i)(x - \sqrt{2}i) = y^3$. This is now an equation in the domain $D = \{a + b\sqrt{2}i: a,b \in Z\}$. We can show that $x + \sqrt{2}i$ and $x - \sqrt{2}i$ are relatively prime in D, and since their product is a cube, each must be a cube—in D. (Such a property holds in Z and one must show that it also holds in D.) In particular, $x + \sqrt{2}i = (a + b\sqrt{2}i)^3$. Simple algebra yields $x = \pm 5$, $y = 3$. Of course it is easy to see that these are solutions of $x^2 + 2 = y^3$. What the above shows is that they are the *only* solutions.

The Fermat equation $x^3 + y^3 = z^3$ can be dealt with similarly: $z^3 = x^3 + y^3 = (x + y)(x + yw)(x + yw^2)$—an equation in the domain $E = \{a + bw: a,b \in Z, w \text{ a primitive cube root of } 1\}$.

There is, of course, considerable work to be done in justifying the "details" involved in the solutions of the above Diophantine equations. In particular, we need to introduce the notions of unique factorization domain (UFD) and Euclidean domain, and to discuss some of their arithmetic properties. The equations can be solved in the indicated manner *because* the respective domains D and E in which they were embedded are UFDs.

PROBLEM III: Can we trisect a 60° angle using only ruler and compass?

This is one of the three famous classical construction problems going back to Greek antiquity. It deals with relations between geometry and abstract algebra, and it gives rise to the concepts of field and vector space. It is a standard problem, usually given as an application of Galois theory. I put it centre-stage as a means of providing a "gentle" introduction to fields.

The initial key idea was the translation of the (geometric) problem into the language of classical algebra—numbers and equations. This occurred in the 17th century. Thus the basic goal became the construction of *real numbers*, often as roots of equations. How do fields and vector spaces enter the picture?

If a and b are constructible, so are $a + b$, $a - b$, ab, and a/b $(b \neq 0)$—all this is easy to show. Thus the constructible numbers form a field. But what are they?

If a is constructible, so is $\sqrt{a}$. We can therefore construct the field $Q(\sqrt{a}) = \{p + q\sqrt{a}: p,q \in Q\}$. This introduces the important notion of *field adjunction*. The objective is to show that *all* constructible numbers can be obtained by an iteration of the adjunction of square roots. For this the classical algebra of the 17th century is insufficient—we need the modern algebra of the 19th century: fields and vector spaces.

Payoffs:

(a) A characterization of the real numbers as a complete ordered field.

(b) A discussion of algebraic and transcendental numbers.

(c) A characterization of finite fields.

(d) Proof of a special case of Dirichlet's theorem on primes in arithmetic progression, namely that 1, 1+b, 1+2b, 1+3b,...contains infinitely many primes. To show this we need cyclotomic field extensions.

(e) Are complex numbers unavoidable in the solution of the so-called irreducible cubic? (This is an equation of the form $x^3 + ax = b$, irreducible over Q, in which all three roots are real.) The answer is "yes." There is a proof using the considerable power of Galois theory, but the result can be established by means of elementary field-extension theory.

PROBLEM IV: Can we solve $x^5 - 6x + 3 = 0$ by radicals?

Problems such as this, dealing with the solution of equations by radicals, gave rise to Galois theory. They touch on the relations between classical and abstract algebra.

Galois theory, in its modern incarnation, is a grand symphony on two major themes—groups and fields, and two minor themes—rings and vector spaces. Galois theory is thus a highlight of any course in abstract algebra. But to do it in detail would take most of an entire term. Moreover, the proofs of theorems are often rather long (and sometimes tedious), and the payoff is long in coming. The intent in this course is to get across some of the central ideas of Galois theory—for example, the correspondence between groups and fields and what it is good for—often with examples rather than proofs.

PROBLEM V: Papa, can you multiply triples?

This problem deals with extensions of the complex numbers to hypercomplex numbers, for example the quaternions. (The question in the title was asked by Hamilton's sons of their father to inquire whether he had succeeded, after years of effort, in obtaining an algebra of triples of reals analogous to the complex numbers.) The problem bears on relations between arithmetic/classical algebra and abstract algebra, and it gives rise to the concepts of an algebra (not necessarily associative) and a division ring (a skew field).

Hamilton's quaternions—a noncommutative "number system"—was conceptually a most important development, for it liberated algebra from the canons of arithmetic. The history of their invention (in 1843) is well documented and gives a rare glimpse of the creative process at work in mathematics.

Are there "numbers" beyond the quaternions? (What is a number, anyway?) Cayley's and (independently) Graves's octonions (8-tuples of reals) gave an affirmative answer, and raised the obvious question whether there are numbers beyond the octonions. A negative answer this time, given by Frobenius and C.S. Peirce (again independently). Implicit in these ideas are the notions of division ring and algebra.

GENERAL REMARKS ON THE COURSE

(a) The first and last problems (and probably also the second) are atypical in an abstract algebra course, but I have found them to be pedagogically enlightening and rich in algebraic ideas. Historically, they signalled the transition from classical to modern (abstract) algebra.

(b) The first problem begins with a "simple" numerical question. The idea is to ease students gently into the abstractions.

(c) While the sequence of topics in algebra books (and therefore in algebra courses) is usually: groups, rings, and fields, our problems introduce students first to rings, then fields, and finally groups. I have found this order to be more effective. It leaves to the end the conceptually most difficult notion, that of a group ("unnatural" to students).

(d) I have listed only five problems. This does not appear to be sufficient for an entire course, it might be argued. However, the problems are wide-ranging and rich in ideas, and they are extendable in various directions.

REFERENCES

Adams W. W. and L.J. Goldstein (1976). *Introduction to Number Theory*. Englewoods, NJ: Prentice-Hall.

Birkhoff, G. (1973). Current trends in algebra. *American Mathematical Monthly* 80, 760-782.

Birkhoff, G. and S. MacLane (1977). *A Survey of Modern Algebra*, 5th ed. Wellesley, MA: A. K. Peters.

Goldstein, L. J. (1973). *Abstract Algebra: A First Course*. Englewoods, NJ: Prentice-Hall.

Hardy, G. H. and E.M. Wright (1978). *An Introduction to the Theory of Numbers*, 5th ed. Oxford, UK: Oxford University Press.

Kantor, I. L. and A.S. Solodovnikov (1989). *Hypercomplex Numbers*, translated by A. Shenitzer, New York: Springer-Verlag.

Kleiner, I. (1995). The roots of commutative algebra in algebraic number theory. *Mathematics Magazine* 68, 3-15.

McCoy, N. H. and G.J. Janusz (1992). *Introduction to Modern Algebra*. Dubuque, IA: Wm. C. Brown.

Pollard, H. and H.G. Diamond (1975). The Theory of Algebraic Numbers., *Mathematical Association of America*.

Pycior, H. (1981). George Peacock and the British origins of symbolical algebra. *Histtorica Mathematica* 8, 23-45.

Richman, F. (1971). *Number Theory: An Introduction to Algebra*. Belmont, CA: Brooks/Cole.

Stewart, I. (1975). *Concepts of Modern Algebra*. Hammondsworth, UK: Penguin.

Tignol, J. P. (1988). *Galois Theory of Algebraic Equations*. New York: Wiley.

van der Waerden, B. L. (1985). *A History of Algebra*. New York: Springer-Verlag.

van der Waerden, B.L. (1976). Hamilton's discovery of quaternions. *Mathematics Magazine* 49, 227-234.

Topic Session 2

ALGEBRAIC UNDERSTANDING

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TOPIC GROUP DESCRIPTOR

The lusted after "mathematical understanding" of the education system continues to elude us though we all presume, or hope, we are teaching for "it". This topic group began with a brief account of my own quest for an educationally useful model of mathematical understanding in a key area of school mathematics, algebra. This four year project involved twenty-five people from six countries—education researchers, mathematicians, teachers and students—and their contributions through interviews (many at past CMESGs), follow-up discussions, and references to the literature were extremely rich.

After a brief presentation of the multi-metaphoric model for algebraic understanding that emerged in this study, two of the metaphors were examined to demonstrate the understandings (including the debates and misunderstandings) of algebra and algebraic understanding that are brought to light by the various metaphoric perspectives.

Since a number of participants in the topic group had been involved in the project at some point over the four years, it was an occasion for me to present some of the results and launch questions for further research concerning the viability of the model, its usefulness, degree of applicability to other areas of mathematics and its links to other models of understanding.

THE UNDERSTANDING SAGA BEGINS

The starting point for this research was my own frustration with the use of the word "understanding" in the field of mathematics education and most particularly in my field of research: the learning and teaching of algebra. The word was ill-defined yet increasingly peppered throughout education documents; it was mystified and tended to be associated with some inherent "thing" that some students had and some didn't. Yet it was being used to discriminate and exclude certain students and groups of students who, it seemed, did not "have it." I felt that there was little correspondence between what was being said in the literature concerning understanding and what I witnessed in the groping and growing understandings of my students.

The available research tended mainly to explore and compile long lists of students' misunderstandings (as surmised from their errors), to create hierarchies or dichotomies of understandings, or to explore the philosophical foundations of understanding. All of this work contributed to the increased mystification and, to use a now familiar word, "reification" of the process of mathematical understanding. Like the "Emperor's new clothes," understanding had become a thing—and, moreover, measurable.

THEMES EMERGE

Four and a half years ago, that malaise concerning algebraic understanding fused into a research project which I decided to undertake in the context of a doctoral program. It was clear from the start that my research question, What does it mean to understand algebra?, ought to be addressed to the community which uses and gives meaning to the word "understanding" in mathematics today—a community that has taken on international dimensions. I decided to involve twenty-five people from the international mathematics education community in the study: four mathematicians[1], four teachers of high school algebra[2], four exceptionally articulate students from Canada and the USA, and thirteen mathematics educators/researchers[3].

After four years of a recursive process of interviews, analysis of protocols, follow up in the literature, meetings and email discussions with the group of twenty-five, seven themes emerged which allowed me to group everything that had been said about algebra and algebraic understanding. These themes were based on very different views of algebra—and were, to varying degrees, metaphorical in nature. The themes provide both a model[4] of algebra as well as a model of algebraic understanding. The seven perspectives on school algebra on which the model of algebraic understanding was built were: algebra is a tool, an activity, a way of thinking, a culture, a generalized arithmetic, a language, and a school subject.

The richest of the seven were the language and activity metaphors for algebra with understanding involving fluency in the language and engagement in the activity. Algebraic understanding as a particular way of thinking and as expertise with the algebraic tools were also very productive metaphors. Although the metaphor involving generalized arithmetic drew considerable criticism, it too brought with it many reflections that might not have surfaced otherwise. Algebra as a school subject is probably the least metaphorical unless one considers it as being *like* (and unlike) any other school subject. Standing outside algebra in this sense and seeing its insertion in the school and society did afford other learnings and certainly the international differences—particularly those involving the place of functions in algebra—were very important in showing the cultural differences in definitions of and attitudes towards both algebra and understanding. And finally, algebra as a community or culture, with understanding as belonging to (or insertion in) that culture, linked with the other metaphoric themes as well as bringing some perspectives of its own.

These seven themes have been called metaphoric here. Without taking a position on the question of whether all our understandings—as, for example, Lakoff and Johnson (1980) would have us believe—are metaphorical, certainly the responses to the question concerning algebraic understanding were. The nature of these responses tended to be of the form: "Algebra is X and therefore, algebraic understanding is Y" where Y was constructed around X. Awareness of, and belief in, the metaphoric nature of statements like "Algebra is X" varied from person to person. As well, it was clear that the metaphorical referent, X, was perceived very differently by the various interviewees and others referred to in the literature. Thus it is necessary when considering metaphoric expressions to keep in mind the X being referred to can be a very different thing according to the experience of the person speaking. One does, in fact, learn a considerable amount about the speaker's view of and feelings about X. But rather than continue a discussion of the metaphoric nature of the themes, let us examine two of them—the tool and language metaphors—in order to illustrate the nature of the model and get some feel for the type of learnings each brings or, to speak metaphorically, the light each of the perspectives throws on algebraic understanding.

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[1] Of these, three are familiar to CMESG members: David Henderson, Bernard Hodgson, and Peter Taylor.

[2] George Gadanidis, a CMESG regular was among these.

[3] Many of these are familiar either as invited speakers or members of CMESG: Michèle Artigue, Jere Confrey, James Kaput, John Mason, Richard Noss, David Pimm, Susan Pirie, Alan Schoenfeld, Anna Sfard.

[4] I am using the word "model" here in Chevallard's sense: a machine to think with. The 7 metaphor perspective on algebra and algebraic understanding has been helpful to me in thinking about and analyzing educational literature, school curricula and textbooks, algebra classrooms, and the general discourse on school algebra within the school culture.

UNDERSTANDING ALGEBRA AS A TOOL

The most obviously metaphoric of the themes for algebra was "algebra is a tool."

> Algebra is *a tool for life*. Algebra is *a tool for modeling, mathematizing, and meeting empirical needs. It's the penknife of science. It's one complex tool with all these little things that come into play. Very much like a pen knife would have a blade with a little screw driver on the end, algebra is the same.*

Those who used this metaphor mentioned a wide variety of tools—hammer, screw driver, knife—and very different users of tools from technicians, those who follow orders, to creative artisans. Very different attitudes—rom very negative to positive—towards tools and their users seemed to underlie what was being said. Yet both produced a number of learnings about algebraic understanding.

From the negative bias towards tools we saw the dangers of the algorithmic side of algebra, how much of algebra can become mechanized, automated, divorced from thought, how users can become drudges, or too connected to "reality" to "fly" and see the beauty of mathematics. We see the secondary or service role of algebra where its users only serve to point to, or are the handmaidens of, the real citizens of the mathematical realm. Algebra can be perceived as a service activity (in the service of other mathematics and mathematics activity) but meaningless in itself. Algebra here is for non mathematicians, beginners, the masses; it is neither beautiful, structural, nor aesthetic; it meets empirical needs but is not intellectually satisfying.

Algebraic understanding from this perspective is either an oxymoron (i.e., there is nothing to understand) or involves only the low level of understanding involved in knowing when and how to use the algebraic tool or tools. Another opinion was that understanding must precede algebra (i.e., arithmetic understanding) and follow it while algebra itself is skipped over.

From the positive perspective, we saw algebra is a tool that gives the user mathematical power and precision, helps in solving problems in science and life, makes our thinking more effective, carries and transforms messages. Users of the algebraic tool can become highly skilled artisans or crafts people, creators, artists. Here algebraic understanding involves being able to handle the algebraic tool with the flexibility, dexterity and sense of purpose of the artisan.

The negative attitudes towards tools surfaced again in discussion of the tool/object dichotomy. Whether or not it was expressed as a tool/object or process/object dichotomy, there was a tool/other nature to mathematics and, for most, to algebra. And in nearly all cases, it was the "other" that was valued.[5] I summarized in this way:

> The tool is for the masses while the "other" is for the elite; the tool is for high school while the other is for university; the tool is useful, its other is beautiful. There are users of tools whose "others" are creative craftsmen. Tools are empirical while their "other" involves abstract thinking and understanding. (Lee, 1997, pp. 129-30)

In the course of discussion around the tool/object theme, it became clear that there is a major cultural difference between what the French call the "outil/objet" or tool/object dialectic (coined by Douady) and what Sfard and others call the process/object dichotomy. In fact the former's tool seems to be the latter's object. While the dialectic movement between the two (tool <--> object; process <--> object) is recognized by all, it is considered from a sociological perspective by the French (the social process of the institutionalization of knowledge) and from the individual psychological perspective by Sfard and others. Yet there appears much is to be gained from looking at the "thing-ifying of tools" (and possibly the tool-ifying of things) from both of these perspectives.

This leads to two very different perspectives on algebraic understanding. The first is a psychological process which is often seen as a single moment when the tool or process becomes an object in the mind (reification). Understanding is also seen as the ability to move back and forth between the two perspectives (process <—> object, or tool <—> object). The second view of algebraic understanding is more social than psycho-logical. Since objects, and the relationships between them are socially defined, to understand algebra is to have those relationships to the tools and objects of algebra that are defined and sustained by one's institution (class, school, school system,...).

UNDERSTANDING ALGEBRA AS A LANGUAGE

This second metaphor for algebra was perhaps the most ubiquitous and entrenched—not only among my research group of twenty-five but in the research and school literature. For some, algebra IS a language with no metaphor involved.

> *You can think of school algebra as a language, notational framework, means for abstracting. It's very hard to abstract if you don't have a language with which to abstract.*
>
> *You see, I find it difficult to objectify algebra as a thing in this sense. I don't think it is a thing. It's more like a language than anything else.*
>
> *Probably the first thing I almost said was that algebra is a language.*

Everyone has some experience of language which is probably why this metaphor is so universally appealing. There is, in the shared experience of language—particularly in relation to one's mother tongue—quite a list of features of language which makes discussion of algebra under this metaphor extremely rich. The generally acknowledged aspects of language that came out in my interviews and in the literature included:

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[5] We came face to face with a paradox in our discourse (particularly that of the reform movement) which would have algebra move away from the old "rote," mindless symbolic work of the past and use algebra as a problem solving or modeling tool. Yet the very denial of the old, means that students cannot work on the models they have built in order to produce new knowledge—or simply to answer the question posed in the problem. (The latest North American answer to this dilemma seems to be to propose numeric and graphic tools offered by the new technologies and to re-name the latter "algebraic" because they are being used to solve problems that were formerly solved by algebra.)

- its "naturalness" (learned easily and early and without any particular teaching—the "language instinct")
- a certain automation or fluency is necessarily achieved
- it has a grammar or syntax, a semantic or meaning aspect, as well as functions, aims, or purposes
- it has both written and spoken form
- it serves for communication and thought (the links between language and thought are strong)
- it is best learned in a milieu where it is the major means of communication (immersion or enculturation)
- second languages are much more difficult to learn than first languages (though early introduction is once again most conducive to learning)

Discussion of algebra as a language generally began with one's mother tongue as the source of the metaphorical inspiration. The aspect of "naturalness" (the un-naturalness of algebra) led most to switch the metaphorical source to a second language and their experience of that. This is where a marked difference in the language metaphor appeared since very few people have a shared second language experience. International differences here were marked. For instance, Europeans tended to have a very rich, early and broad experience of second language (exposure to several other languages from a fairly early age). Americans, and to a lesser extent Canadians, often referred to their considerably later experience of learning a computer language in using the second language metaphor. These very different experiences (learning a spoken language of a region vs. learning a way of communicating with a machine) led, of course, to very different insights about algebra.

Yet considering algebra as a first, second, or other language was certainly productive in many ways. We were reminded of how we learn a language and how algebra might more successfully be taught as a second language profiting from all the new pedagogical insights in the language teaching area: use of the language from the beginning to express oneself, the advantages of immersion, etc. New pedagogies in the field of music were alluded to as well. Research in the field of linguistics was also felt to be rich in lessons for the learning and teaching of algebra.

In as much as the language metaphor for algebra was seen to fit, the lessons for algebraic understanding were many and varied. Among these we find:

- Algebraic understanding, like language understanding grows best in an environment or culture where it is the language of exchange.
- Algebraic understanding neither precedes nor follows work with the letter symbolic but grows in much the same way one's understanding of a language grows when one lives in a culture where it is spoken.
- There are implicit as well as explicit understandings of algebra, formulated and unformulated.

A number of people in the field seemed to have given considerable thought to the semantic and syntactic features of algebra and this dichotomy wove in and out of discussions even when the language metaphor was not being consciously used. The syntax of algebra was seen to be the rules for manipulating symbols and the semantics of algebra was associated mainly with its referent meaning outside algebra itself (in other mathematics or the real world). When discussion on this was deepened there was some acknowledgment that algebra has an internal semantic as well and that without this symbolic manipulation is completely haphazard and unsuccessful. In fact, several people referred to the "semantic fields" of algebra, as well as to its social, local, and shared meanings. Thus the business of student involvement in "empty syntax" or mindless symbol juggling is questioned and the dynamic interplay between syntax and semantic in algebra is put forward for consideration. We are reminded that algebra requires the learner suspend or ignore extra-algebraic semantic fields and concentrate on the "semantics of the syntax" in order to make headway in algebra. From this perspective, trying to constantly link algebraic manipulations to their "real world" or even mathematical referents is counter productive for the student and defeats the whole achievement of algebra.

In fact one can sum up algebraic understanding under this metaphor as mastery of the algebraic language, its forms, its functions, its meanings. To understand its forms involves expertise in symbol manipulation, knowing the conventions of the algebraic language and achieving a certain level of automation. To understand its functions is to bring into play the wide range of its purposes or objectives and to understand its meanings involves mastery of the semantic fields (internal and external) of algebra.

Here the metaphorical misfit between understanding algebra and understanding a language-led to the expression of a number of other insights into the nature of algebraic understanding. Unlike a language, algebra provides its own meanings without any external referent while knowledge of the algebraic language automatically implies having a meta-awareness as well. Two kinds of algebraic understanding were, in fact, identified: external (where symbols refer to non-algebraic objects as in natural language) and internal (where the meaning of symbols comes from within the algebraic symbolic system. The special features of algebraic understanding that are not shared with any language are firstly this recursive, self referent, internal aspect and secondly the manipulative action aspect of algebra, algebra as "something you do".

The manipulative or action side of algebra that did not fit well with the language metaphor led some to try combining or linking the tool and language metaphors and to speak of algebra as a language tool. Others moved on to exploration of another metaphor for algebraic understanding based on algebra as an activity.

CLOSING WITH QUESTIONS

We did not have time in the topic group session to explore this or the four other metaphors for algebraic understanding nor will we do so here. It is hoped that the two metaphors we have outlined give a sufficient hint of the nature of the multi-metaphoric model to allow for further discussion in the course of future meetings, for the reformulation of questions concerning algebraic understanding, and perhaps for an attempt at formulating a new metaphor for algebraic understanding that will take us into the next century. A second hope is that both the research method and the model might be taken into other areas of mathematical understanding. And finally, it is hoped that this multi-metaphoric model with all its richness and conflict will push the mathematics education community to be more wary of its use of the word "understanding" in relation to all fields of mathematics.

REFERENCES

Lakoff, G. and Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: The University of Chicago Press.

Lee, L. (1997). Algebraic understanding: The search for a model in the mathematics education community. Unpublished doctoral dissertation. Université du Québec à Montréal.

Topic Session 3

STUDENT EXPLANATIONS IN COLLEGE LEVEL COURSES

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COMPUTATIONAL AND CONCEPTUAL QUESTIONS

Students completing a first linear algebra course will probably be able to bring a matrix into reduced echelon form—otherwise they are likely to fail the final course examination. Consider, however, what they might answer to questions such as

- Can any matrix be brought into reduced echelon form? Why or why not?
- What does the reduced echelon form say about the rank of a matrix?
- Would you typically want to bring into reduced echelon form the coefficient matrix of a system of linear equations or its augmented matrix? Why?
- How and why can you tell from the reduced echelon form of an augmented matrix how many solutions a system has?
- Is the reduced echelon form required for finding out how many solutions a system has?
- If not, what is the reduced echelon form required for?

In the above example, the mathematical content, linear algebra and, more specifically the reduced echelon form for matrices, is to a large extent incidental. Analogous examples could have been chosen from other topics in linear algebra, from calculus or from most other college or introductory university mathematics courses. On the other hand, the frequency with which the question "why?" appears in the list of questions is symptomatic rather than incidental. It is intended to draw the reader's attention to non-computational, conceptual questions which require an explanation or justification from the student.

In typical college mathematics courses for, say, future engineers, students are expected to learn how to solve a certain set of problems and demonstrate that skill in exercises and examination papers. The large majority of these problems are computational. Often, they consist in choosing the correct formula or procedure and carrying out the corresponding calculations. In most cases, these calculations could just as well be carried out by a computer algebra system; and often, they are given more weight in examinations than the choice of the correct formula; and conceptual questions, if they appear at all, are considered a bonus.

Recently, some college textbooks and teachers have begun to require students to write explanations for their computations; for example, Lay's (1994) textbook includes writing exercises; Lay states that such exercises are included because scientists and engineers need to be able to say precisely what they mean and why something is true. Similarly, Barnett (1992) has asked her calculus students in exercises and in tests to argue the truth or falsity of statements such as 'If f is differentiable at a, then f(a) is defined' or 'If f is not differentiable at a, then the graph of f has a vertical tangent at a'. She used such problems to replace formal proof questions; she found them to be quite a challenge for her students, and states that they are appropriate to test students' understanding of theoretical points including the logical form of theorems.

It is to be expected that technical aspects become less and less important in our teaching (not least because of computer algebra systems), and that conceptual aspects become correspondingly more important. The question thus arises which kind of relationship between computation and reasoning college courses should aspire to.

Apparently, college teachers who ask their students to explain do so out of a general feeling that the students will acquire better skills of communication and understanding. Schurle (1991), for example, asked his students to write because 'computational problems do not show whether a student *really understands*'. His data show that students feel writing assignments improve their understanding. He neither gives a detailed analysis of his aims, nor does he develop criteria for what constitutes a satisfactory, a good or an excellent explanation.

It was the purpose of the session I led at the 1997 CMESG conference to raise the consciousness of the participants for the necessity of reflecting on such criteria. It is the purpose of this paper to report on the session and on some criteria which were proposed there and elsewhere.

NON-COMPUTATIONAL QUESTIONS, STUDENTS AND TEACHERS

Most students do not particularly like conceptual questions which require answers to "why?" questions nor are they especially good at them. For example, the following two problems were both assigned during the same week from the same textbook (Anton, 1994) to the same class of 57 students as homework assignment in the framework of a course I recently taught:

P1: Show that the following set of vectors is linearly independent: $\{(-3, 0, 4), (5, -1, 2), (1, 1, 3)\}$.

P2: Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

Twenty students handed in the assignment; all twenty solved P1 by reducing an appropriate 3 by 3 matrix; four made computational mistakes; out of the remaining sixteen, eleven explicitly concluded their work with a sentence like "the vectors are linearly independent"; none, however, addressed the question why their computation shows that the given vectors are linearly independent. From among the same twenty students, fifteen omitted P2; out of the remaining 5, two started with "Let's assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent ..."; none of the other three answers was satisfactory either.

More generally, my observations over many years show that, in introductory calculus or linear algebra courses in which explanations are required and contribute to the course grade, four types of student behavior can be observed; typically, each behavior characterizes roughly a quarter of the students. One group of students understand what they are expected to do and start explaining with fair success; another group carefully describe their computations but neither show why they do them, nor argue the truth or falsity of statements; a third group put some more or less meaningful words in between their computations 'because the teacher wants words'; and a last group, just present the computations as they are used to do from high school. It thus appears that a majority of students either do not have the abilities required to explain their computations and answer conceptual questions or they do not use this ability in mathematics courses, even when it influences their grade!

Some of the reasons for this situation have been analyzed elsewhere (Dreyfus, 1997). It appears to be partially due to the nature of mathematics, but more substantially to students' views of and experiences with the nature of mathematics. These, in turn, are based on the education they received. The task thus arises for teachers to communicate their expectations in this respect to their students.

This is not an easy task; textbooks are a case in point. Problems P1 and P2 above, for example, start with the words "show that"; the same textbook has many other questions which start with "show that ..." (as in 1 above, presumably meaning "compute"; or as in 2 above, presumably meaning "prove"), others start with "prove that ...", or with "show by example that"; and still others require students to justify, explain (in some cases "by inspection"), verify, or prove a theorem in a special case. The different intention underlying these formulations are not spelled out and are presumably far from clear to many students.

Most college teachers are excellent at explaining mathematics—it is after all their profession. Moreover, we usually know very well, in any specific case, what the value of a student's explanation is. In any given answer, we can point to what is awkward or missing or superfluous or plainly wrong. We may be less able, however, to provide helpful feedback to the student whose explanations are unsatisfactory or off target altogether. And we may be at a loss to communicate to a class what, generally, is required in conceptual problems and exercises. If we find it difficult to make the criteria we use clear, even to ourselves, how can we take the next step, beyond giving and judging explanations, the meta-step of explicitly characterizing a good, and an excellent explanation! What are the general characteristics we would like to see in students' explanations?

Language is a central feature of explanations. Morgan (1996) has observed that (high school) students found it difficult to produce written texts that were acceptable to their mathematics teachers; she further found it unlikely that the teachers are able to provide advice to the students on how to produce acceptable texts because of their inability to identify the linguistic features of student texts which influence their judgments. She has therefore concentrated on critical linguistic analysis of mathematical texts.

If courses, and students' activity in them are to become less computational and more conceptual, if conceptual problems are to appear more frequently in assignments and examinations, and if student explanations are required more frequently, then it is a central task for teachers to not only assess and grade students' answers to such problems, but also to provide useful feedback of the requirements and criteria which are used in such assessment to themselves, as well as to successfully communicate them to their students. This necessarily includes an explicit clarification of the requirements and criteria.

There are several reasons for the difficulty of this task; one is simply that it is relatively new and, as a community of teachers, we lack experience. Another one is that the question 'what is an explanation?' is closely linked to the question 'what is understanding?' Explanations are given, at least by teachers, to promote understanding. Thus there is a large degree of subjectivity involved: One person understands a particular explanation, another does not. Thus, what is satisfactory to one is not to another.

At least, a teacher's explanation usually has a well defined audience; but what is the audience of a student's explanation given in an assignment or an examination? Does it need to promote understanding? Does it need to convince? Convince as a proof does? Is it sufficient that it convinces the teacher? And of what does it need to convince the teacher? That the mathematical claim is correct? Or that the student is conscious of the most delicate aspects of the problem and their details? What level of detail is appropriate in which circumstances? What makes a certain aspect relevant and another superfluous, even disturbing?

Are there certain characteristics to a 'good' student's explanations? Correctness of the final result is certainly not the main issue. If it were, one would have to accept 26/65=2/5 whether it was justified by an educated guess, by cancellation of the factor 13, or by cancellation of the digit 6.

In a paper written in support of writing as an important part of our students' education, Price (1990) gives some guidelines to his students: correct mathematics, complete but short sentences, balancing words and symbols (about equal amounts of each), using equality signs correctly, using different letters for different objects, defining terms, giving reasons, and answering the question; Price sees the ultimate test as: can someone learn from what the student wrote? The next section will attempt to address the problem more generally, taking into account proposals by a large number of teachers rather than a single one.

CRITERIA

Most, if not all conceptual problems ask students to show, explain, justify or prove something. What criteria are teachers using for judging students' explanations and arguments?

- What general characteristics do acceptable answers have?
- What is the relative importance of these characteristics?
- Can one formulate problem-independent criteria for acceptable student explanations?

In this section, I compiled the outcomes of several attempts at generating criteria, including the one in the session I led at the 1997 CMESG conference. The result is not a well reflected and well structured arrangement of criteria but rather the somewhat edited and partially ordered outcome of repeated brainstorming; it is put at the disposal of the teaching and research community as a basis for further work. The structure within which criteria are presented and commented on below should be considered as preliminary; a presentation of criteria in a multiple connected map structure, similar to a concept map, might eventually be more appropriate than the categorization which I have used here.

Relativity and Subjectivity

Very few of the criteria listed below are general in the sense that they are always appropriate. Criteria depend on the context and situation, on the (declared or undeclared) aims of the course, the teacher, the student, the particular assignment, and so on. With few exceptions, the criteria are thus likely to be appropriate in some cases but not in others. The few exceptions which might have some general validity are marked as "minimum criteria."

The aims of a teacher when requiring an explanation, may range from the evaluation or assessment of students' skills and understandings over a diagnosis of their difficulties in order to plan further instruction to a planned learning experience or peer help for another student. Should an explanation whose aim is student learning have the same characteristics as one whose aim is assessment?

When, as a teacher, I explain something, I explain it to somebody and I usually have a reason to explain it. My explanation will depend on this reason and on the audience. Often, my audience is students and my aim is to help them learn. The students' reasons for explaining something are often far less clear. Their audience may be "friend", "peer", "foe", "self", "teacher", maybe even "mathematician." What is their task? To whom do they explain? And why? Whom do they need to convince? And of what? Do they explain to their teacher, or to a fellow student? Does somebody need to learn from their explanation? Certainly not the teacher, at least not usually. Do they need to convince somebody? Again, usually not the teacher, at least not of the fact that a given mathematical statement is true; but maybe they are supposed to convince the teacher of the fact that they understand a certain mathematical fact or process or relationship. And the teacher may have very subjective criteria for judging such understanding.

Explanations are thus relative to aims, audience and often also relative to a particular course in which the student happens to be enrolled.

Content

Nevertheless, there are some global characteristics which are likely to be valuable in any explanation. They include:

- The relevant topic must be addressed. (Minimum criterion)
- The question(s) must be answered. (Minimum criterion)
- The explanation should show evidence of reasoning rather than be only prescriptive or descriptive.
- The explanation should say why things are done, not only how.
- The explanation should not only say that things are as they are, but why.
- There should be no superfluous digressions.
- The explanation should contain no incorrect mathematics (minimum criterion).
- The underlying reasoning should be correct.
- The result should be correct.
- The explanation should show that the student is conscious of the main points.
- The explanation should show that the student is conscious of delicate junctures in the argument.

Many of these issues are further detailed below. The level of detail required is one of the issues which remains wide open. Some criteria which have been proposed in this connection are:

- Newly learned material (content and process), which is relevant to the course should be addressed. On the other hand, material which has been learned earlier can be omitted. (For example, there might be a need for using some computational procedures with algebraic fractions in a linear algebra course but there is neither a need nor a point in explaining why these procedures are correct.)
- The argument should be convincing; it should not require the reader to fill in gaps.
- The standard symbolism is already an abbreviation and should not be shortened further.

Language and Style

Clarity and issues of language, style and elegance include:

- Problem, examples, and solution should be stated clearly.
- There should be a balance between words and symbols, between the verbal and the formal.
- Short sentences contribute to clarity in mathematical writing.
- Mathematical language, symbols, notations and conventions should be used accurately and correctly.
- The connections between mathematical statements should be formulated explicitly and accurately.
- Use of language (including symbols) should be coherent throughout. Coherence in language is a means for connecting related ideas.
- The intended reader should be able to follow the presentation. One might remark that this is a very vague criterion because in many cases the "intended reader" of a student's explanation is not the teacher but a hypothetical reader who exists in the teacher's mind, and whose ability to comprehend the student's text is lower than the teacher's.
- Jargon should be limited to what is well understood by the student as well as the intended reader.

- Redundancy may be used if it is done consciously and with a purpose. Otherwise, conciseness is preferable.
- Elegance of the argumentation is desirable.

Logic and Generality

Criteria related to logic seem to play a predominant role in every discussion on student explanations. There are two reasons for this. One is that these criteria tend to be most familiar and easiest to formulate for many college mathematics teachers. The other is that at the beginning college level, explanation is often used as substitute for formal proof, and in proofs logical aspects are important. The criteria proposed under the present subtitle thus tend, on the whole, to be somewhat biased toward proof.

- The reasoning should be presented according to a logical development.
- Arguments should be valid (informally logical).
- Statements should be supported by previous statements, definitions, etc.
- Definitions should be used correctly.
- Attention should be paid to the distinction between premise and conclusion, to sufficient versus necessary conditions.
- Attention should be paid to the correct use of if-then and other logical connectors.
- Counterexamples should be used appropriately.
- Jumps in argumentation should be made judiciously; they should reflect transitions which are obvious to the students rather than gaps in their knowledge.

But also:

- The basic idea of an argument may be sufficient for an explanation; an explanation need not be a proof.
- Showing understanding of the concepts under consideration may be more important than establishing a result via formal proof.
- It may be acceptable that the logical flow remains vague, if a clear relationship between premise and conclusion is established.
- If only a special case or a generic example is presented, its generalizability should be addressed.
- Visual or diagrammatic arguments are acceptable if they are supported by appropriate reasoning.
- Numerical examples may not be sufficient.

Connectivity and Coherence

The most complex and probably the most important aspect of explanations is their coherence. Explanations are meant to establish, rely on and make use of connections. Students are thus given the opportunity to show connections they have established between what is to be found or justified and what is given by means of the reasoning they use. In localized, small problems, only local connections can usually be demonstrated. More complex problem situations may elicit more elaborate connections. Larger projects may be even more demanding in this respect. Indeed, it is more difficult to ensure coherence of thought, language, reasoning the more complex a situation is being considered. The following criteria relating to connections and coherence have been proposed:

- The explanation should be internally consistent. (Minimum criterion)
- The explanation should show that the student understands the task or problem as a whole.
- The explanation should make sense globally.
- There should be a clear and appropriate manner to introduce and conclude the presentation.
- There should be a logical argument for the entire problem.

- The argument should be valid and coherent.
- Task-appropriate examples should be chosen.
- It should be made clear why a particular way was chosen to solve the problem or present the reasoning.
- Coherence in language is a means for connecting related ideas.

CONCLUSION

College students are increasingly being required to explain rather than just carry out their mathematical activity. They are thus given opportunities to demonstrate their understanding of the reasons for doing a particular computation and for doing it in a particular way. The underlying assumption seems to be that this contributes to the depth of their understanding and their power to use mathematics. The requirement to verbalize thus contributes to moving students from a descriptive to a justificative mode, from empiricism to rationalism, from a purely pragmatic to a more intellectual view of mathematics (Balacheff, 1987).

This is a crucial transition for the students and we should not expect it to be easy for them. Teachers may support this transition by:

- Regularly requiring explanations,
- Giving many reasoning tasks,
- Giving the result of a computation together with the problem,
- Giving sample explanations to chose from,
- Giving sample explanations to critique, etc.

We should not expect success in this undertaking, though, unless we manage to clarify our goals to ourselves, which is difficult, and then to our students which is even more difficult. In this paper, I have attempted to make a contribution to the process of clarifying these aims. My aim was not to establish criteria, certainly not generally valid criteria, but to sensitize the reader to the need for making criteria explicit and to propose some possible criteria.

Every course, even every didactic situation will have its own set of criteria, just like different subfields of mathematics, different journals, and different periods in history have different sets of criteria. This relativity, however, does not release the teacher from the responsibility to analyze the space of possible criteria, to become conscious of the criteria which are being used, to define these criteria for him- or herself with a suitable level of precision, to interactively constitute with the students what counts as an acceptable mathematical explanation and finally, to properly apply the established criteria to the students' work.

REFERENCES

Anton, Howard (1994). *Elementary Linear Algebra*. New York: Wiley.

Balacheff, Nicolas (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics* 18, 147-176.

Barnett, Janet Heine (1992). Writing assignments in beginning calculus courses. Presentation at ICME-7; Québec, Canada.

Dreyfus, Tommy (1997). Why Johnny can't prove. Preprint, available from the author.

Lay, David C. (1994). *Linear Algebra and Its Applications*. Reading, MA: Addison-Wesley.

Morgan, Candia (1996). The language of mathematics: towards a critical analysis of mathematics texts. *For the Learning of Mathematics* 16 (3), 2-10.

Price, J. J. (1990). Learning Mathematics Through Writing: Some Guidelines. *College Mathematics Journal* 20, 393-401.

Schurle, Arlo W. (1991). Does Writing Help Students Learn about Differential Equations? *Primus* 1(2), 129-136.

Yackel, Erna and Cobb, Paul (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* 27 (4), 458-477.

Topic Session 4

MATHEMATICS TEACHING—HOW IT COULD BE DONE

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I think one of the crucial reasons why our system of education, including mathematics education, is in a critical state is that the profession itself has only a limited, and probably biased idea of what goes on in alternative educational systems. We all could do better by being informed about how things are done by others.

In the past 15 years we have experienced a flood of self-appreciating comments from educational professionals in Canada. Results of the Second International Mathematics Study were frequently reported in a rather biased way (in using means, or medians) that favoured the Canadian results hence illustrating the "relative strength" of Canadian education. Not only were the bases of statistical comparison of the Study very different in the various countries, but the way of testing—multiple choice questions—could favour one group. While in North America testing is mostly done in this (multiple guess) way, this is done infrequently in some other countries. The choice of curriculum, in my opinion, also favoured the North American students. Had the test been based on, say, the Hungarian curriculum, the ranking would have been quite different. Since then I have found other comparative studies with serious inherent bias. One of them compared how much science (including mathematics) is taught *in the last year of secondary studies* in different countries without paying attention whether the same, or more, is taught in earlier years in other systems.

This is why I found it important to objectively show you the apparent difference in educational approaches, expectations, quality of education (and teacher education) by showing you some of the problems students have been expected to deal with at the grade six level in Hungary. In the following, I present several (adapted) problems from a grade six Hungarian work-book [1]. As you will see students at that grade level are introduced to number theory, equations and inequalities, Descartian and polar coordinate-systems, and their transformations, two and three dimensional geometry, functions, optimization and logic, and it is done in an exciting and entertaining way. (The philosophy of education used to base this approach on was developed by Otto Varga, D.Sc., and is called "explorative mathematics".) Along the way students are actively involved in the development of thinking skills as will be illustrated. *All* public schools in Hungary used the same textbooks as prescribed by the Ministry of Education at least until the early 1990s. Their usage was compulsory. At each grade level a work-book was also published for take-home challenge problems. All pupils were expected to participate in solving such problems. From grade five there is subject teaching, and most teachers from grade five up have, in their respective fields, the equivalent of a masters degree in Canada.

Other ways to compare students are also possible. You could compare the amount of homework the average student needs to do in various countries [3] as measured by time; their respective results in international mathematics competitions; or the university entry exams in mathematics.

Originally I intended to speak also about the socio-political reasons why the present "progressive" philosophy has been the ruling ideology for such a long time. Instead, due to limited space, I list some recent critical works in the references.

SELECTIVELY CHOSEN EXERCISES FROM [1]

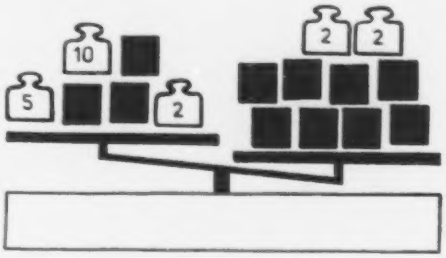
Note: In my talk some 40 problems were presented. Due to the limited space here only 25 will be shown.

When some of these translations were made in the early 1980s Dr. W. R. Allaway of the School of Mathematics, Lakehead University, worked with me, and most of the initial work was done together. We intended to publish the translations, but for different reasons, we were discouraged to finish our work. My comments, if any, are put in square brackets after each problem.

1. The population of Thunder Bay in 1991, rounded to the next thousand, was 114 thousand. What might have the population of Thunder Bay been in 1991?
   Check off with a blue pencil those statements which give only possible answers; with green those which give all possible answers; with red those which give just all the possible answers.

   113 500 $\le$ population of Thunder Bay $\le$ 114 499
   The population of Thunder Bay $\le$ 115 000
   The population of Thunder Bay was surely more than 11 ,000
   The population of Thunder Bay was not larger than 113 000
   The population of Thunder Bay could not be larger than 114 500
   The population of Thunder Bay fell between 113 000 and 115 000
   The population of Thunder Bay was exactly 114 000
   The population of Thunder Bay was larger than 115 000
   The population of Thunder Bay was between 114 000 and 114 499
   The population of Thunder Bay was less than or equal to 114 500
   The population of Thunder Bay was at least 112 000
   113 500 $\le$ population of Thunder Bay $<$ 114 500

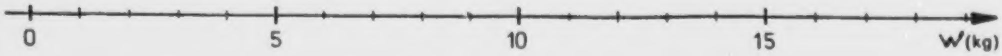
2. Packages of equal weights are measured.
   The position of the scale looks like this:



   Write an open sentence
   on the position of the scale:

   a. Is it possible that the package weighs more than 2 kg? __________
   b. Is it sure that the package weighs more than 2 kg ? ____________
   c. Is it possible that the package weighs more than 1kg? __________
   d. Is it certain that the package weighs more than 1kg? __________
   e. Is it possible that he package weighs less than 5kg? __________
   f. Is it possible that the package weighs less than 4.5kg? _________
   g. Is it possible that the package weighs less than 2.5kg? _________

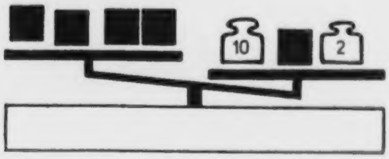
Indicate on the number line the limits within which the weight of the package must be:



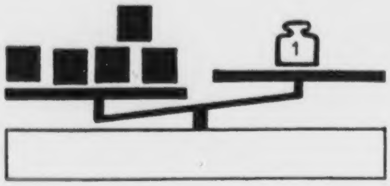
[Note that the underlined question makes the pupil synthesize his/her previous thoughts!]

3. We take two measurements. The positions of the scale are as follows:

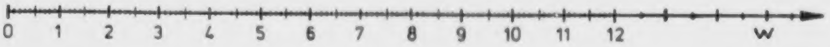
First time:



Second time:

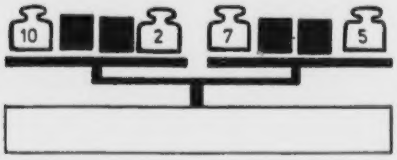


Indicate on the number line the limits within which the weight of the package must be.



4. a. The scale is in balance.
Write an open sentence that
describes the position of the scale.

b. Draw a scale illustrating
the open sentence below.

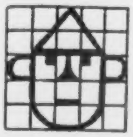


$$\boxed{W\cdot 6+13 = 20+ W\cdot 2}$$

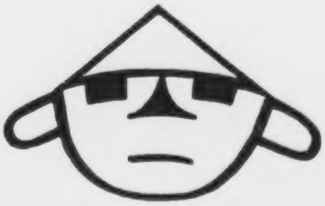
How heavy may a package be?

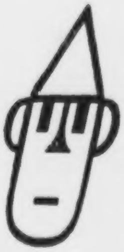
[Here, at first, the student are made to understand the concept of equation and that the weight is well defined. Then, he/she needs to create his/her first equation independently and conceptualize by showing the related scale.]

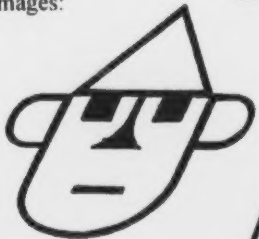
5. Of this picture:

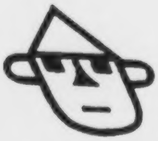


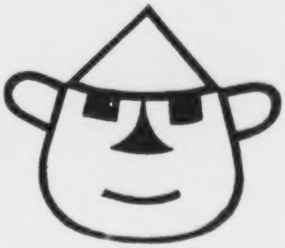
these are the distorted images:

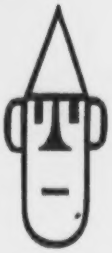


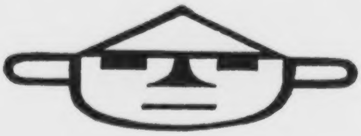






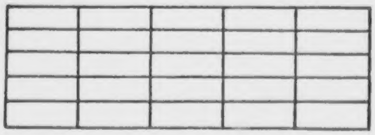




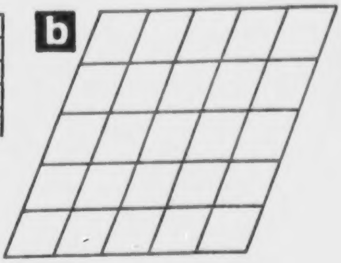


With which grid has each distorted image been made? Write beside each image the letter-name of the proper grid.

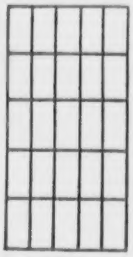
a



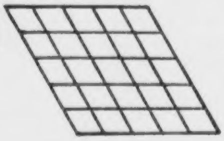
b



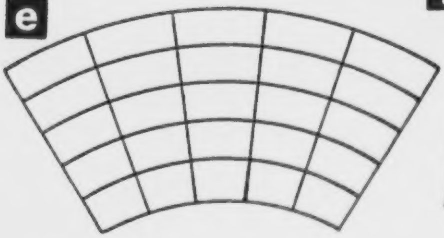
c



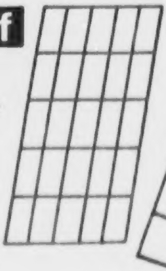
d



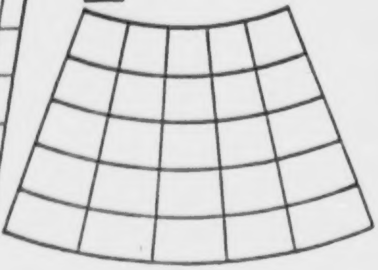
e

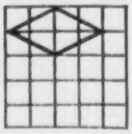


f



g





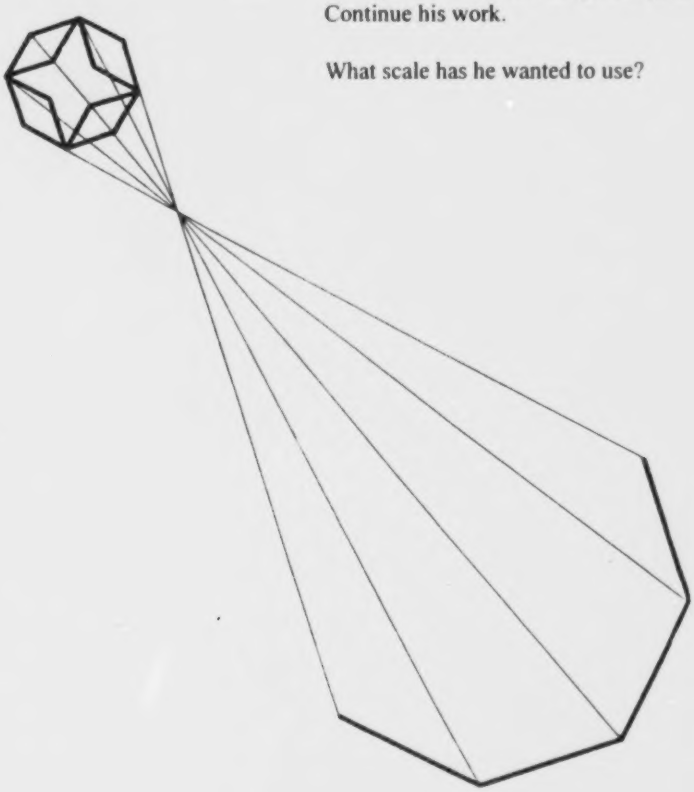
Write here the letter-name of the grid that transforms this rhombus to a proper square:

[First, the problem is to match transformed images with their transformations. Here to create a transformation with a given objective in mind!]

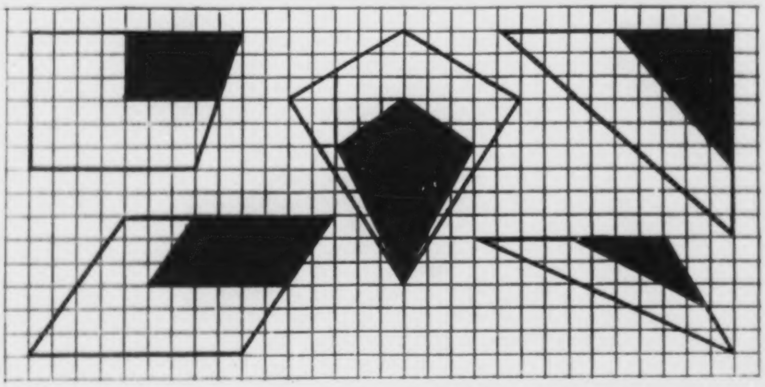
6.

A. Johnny has started to enlarge this pattern.
Continue his work.

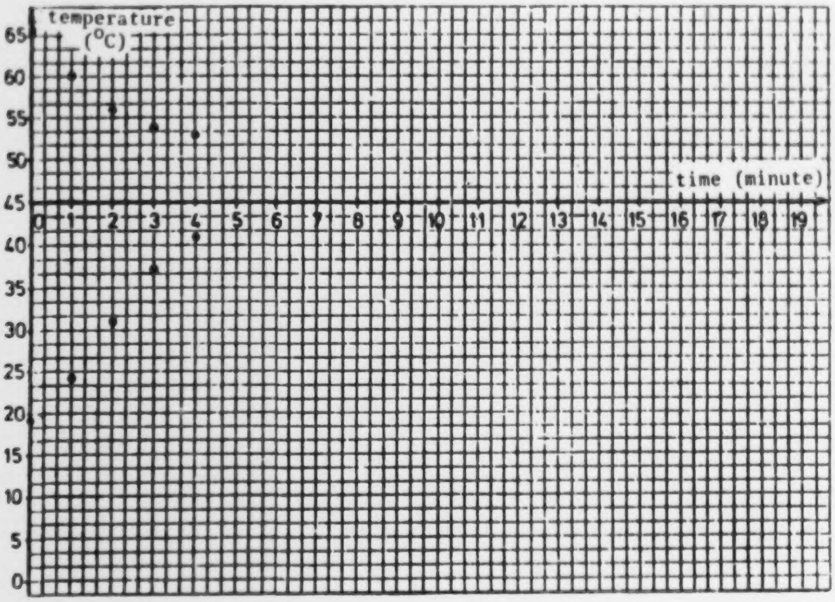
What scale has he wanted to use?



B. Alex wanted to enlarge the black shapes to similar forms, but he erred at each.
Find his errors and correct those.



7. Andrew and Gabriel did this experiment: 500 cm^3 cold water was poured into a glass bowl, and a glass with 100 cm^3 hot water was set in the glass bowl. They measured the temperature of water in both containers.
Then Andrew drew the following diagram after 5 measurements:



A. What do you think the temperature of the cold water was

at the end of the 6th minute ....; at the end of the 7th minute ....;
at the end of the 8th minute ....; at the end of the 9th minute ....;
at the end of the 10th minute ....; at the end of the 11th minute ....?

Write your answers above on the dotted lines.

B. Gabriel wrote up the data he measured:

at the end of minute	0	1	2	3	4	5	6	7	8	9	10	11	12	13
the temperature of the cold water warmed up to (C°)	19	24	31	37	41	43	45	46	46	46	46	45	45	45
the temperature of the hot water cooled off to (C°)	66	60	56	54	53	52	51	50	50	49	48	46	45	44

Continue Andrew's diagram using this data.
[The pupil is made to create a graph based on data.]

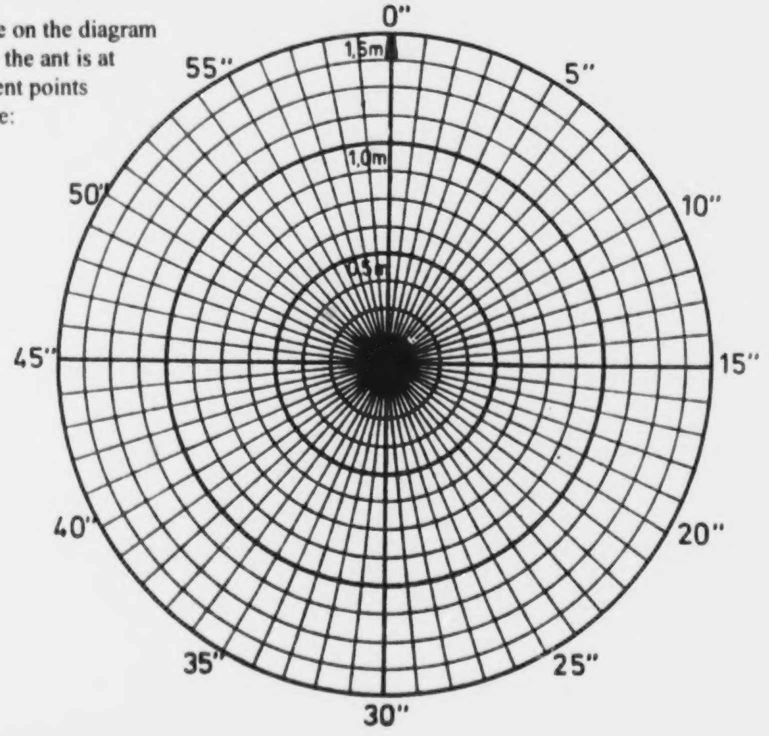
C. From the diagram determine the following:

(1) At the end of what minute was the temperature of the cold water the highest?
(2) During what minute did the temperature of water increase the most?
(3) For how long was the temperature of cold water constant?
(4) At what time did the temperature of the water become equal in both containers?

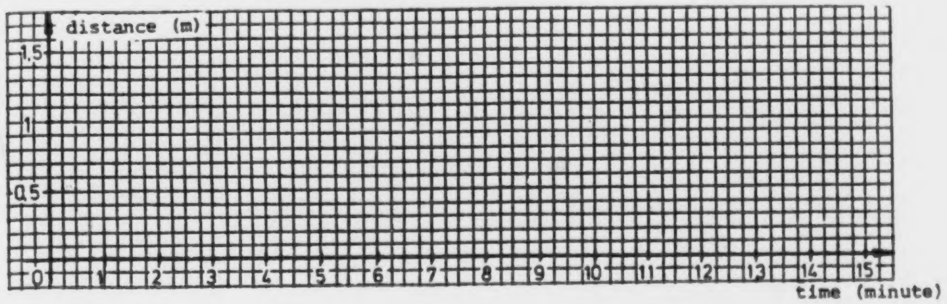
[Here the basic ideas are presented re: maximization, and the slope.]

8. An ant is creeping along the second hand of the clock in a tower. It starts from the middle of the clock and advances by 0.5 m per minute. The hand is 1.5 m long. When it reaches the end of the hand the ant turns back immdediately.

a. Indicate on the diagram where the ant is at different points of time:



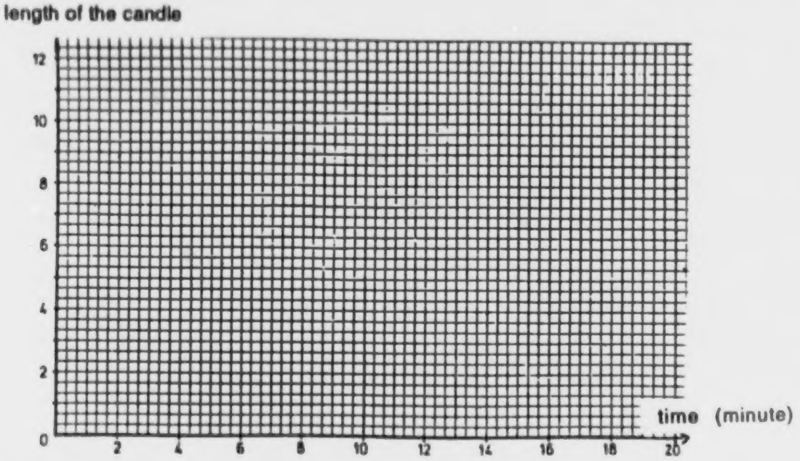
b. Do the same in the rectangular coordinate system:



c. Answer the following questions:

(1) How many times does the ant go to the end of the hand and back in an hour?
(2) How long does it take the ant to get to the end of the hand the first time?
(3) How long does it take the ant to reach the end 15 times?
(4) How long does it take the ant to get around the face of the clock once?

9. Two candles are lit at the same time. One of them is 10 cm long. This will get shorter by 5 mm per minute. The other is 8 cm long. It burns down in 16 minutes. Draw a diagram showing the lengths of the candles.



Now answer the following questions:

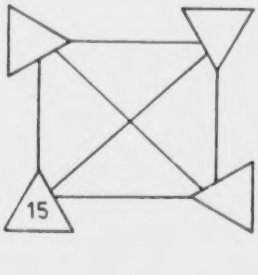
a. When will the lengths of the two candles be the same?
b. When will the second candle be twice as long as the first?
c. When will the first candle be twice as long as the second?

[In this problem students solve equations, –verbal problems–, with the use of graph.]

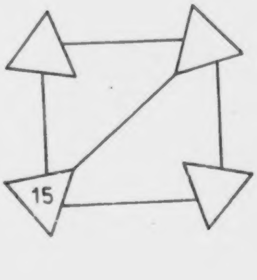
10. Between any two numbers as many lines are drawn as many common divisors they have (not counting 1).

Attempt to find the smallest possible numbers for the blank spaces.

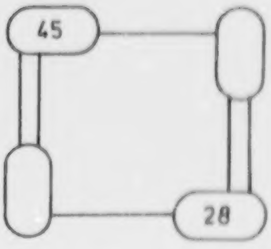
a.



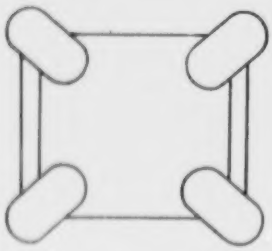
b.



c.

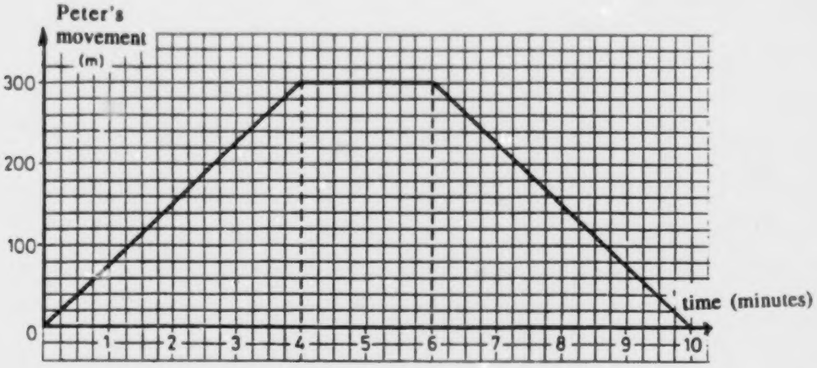


d. Position these numbers in the blank spaces: 99; 117; 44; 52 .



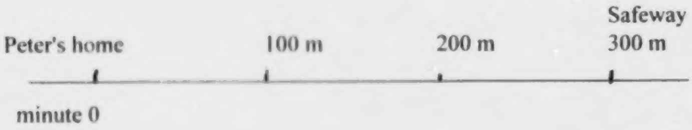
[These are not easy problems for the 11-12 years old! Try to solve them.]

11. Peter went to do some shopping in the Safeway. His movement is illustrated by the graph below. What can you read off the diagram?

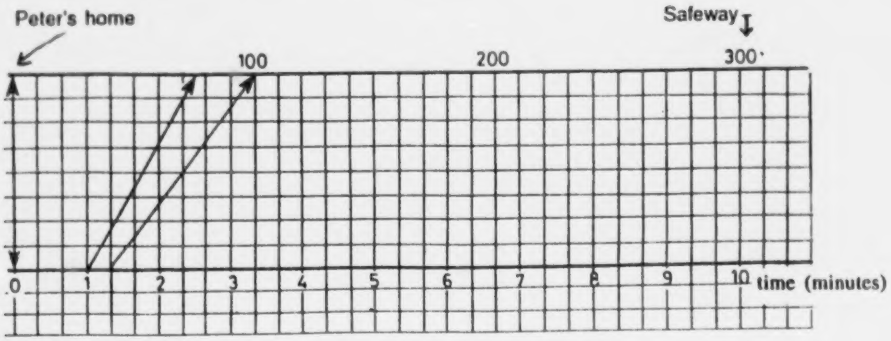


a. How far is the Safeway from Peter's home?
How long did it take to get to the Safeway?
How long did he stay in the Safeway?
On average, how far did he walk in a minute?

b. Peter's home and the Safeway store are along the same road. At the indicated places along the road calculate how long it took for Peter to get there.



c Illustrate Peter's movement also on the diagram below:



12. Ann's mother was 24 when Ann was born. Six years from now Ann's mother will be three times as old as Ann. How old are Ann, and her mother, now? You may try to find the solution here:

Now	mother							
	Ann							
In 16 years	mother							
	Ann							

13. I wrote up a 3-digit number twice. The number 7 is written to the beginning of one of them, and to the end of the other. In this way I got two 4-digit numbers. The first is larger by 2826 than the second number. From the following open sentences which one describes this problem correctly?

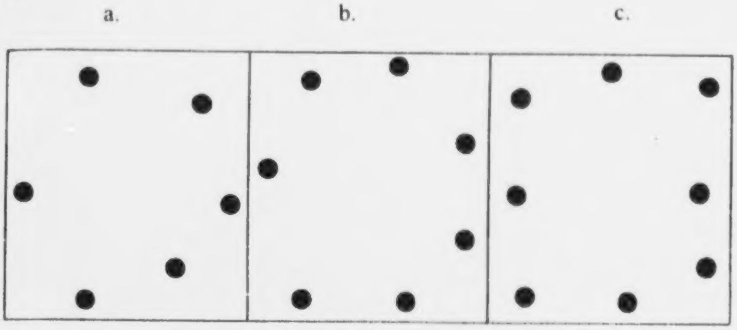
a. $x + 7 = 7 + x + 2826$

b. $10x + 7 = 2826 + 7000 + x$

c. $10x + 7 + 2826 = 7000 + x$

d. $10x + 2826 = 7000 + x$

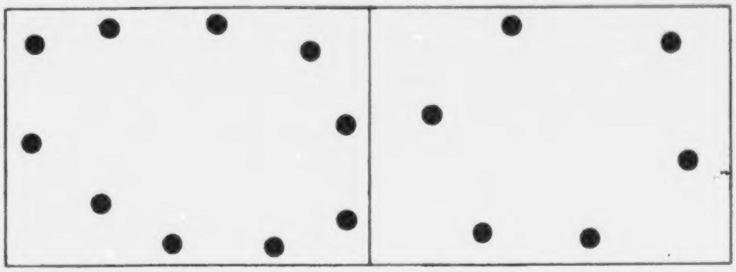
14. Nodes represent towns. Design a road network such that from each town exactly three roads should lead out.


a. b. c.

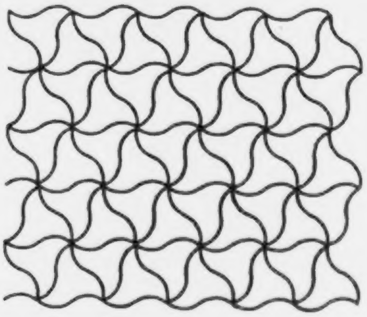
[With reference to problem b.: Note that the student should realize that solving this means to show that there is *no solution* to this problem. Since seven times three is odd, it cannot be divided by two!]

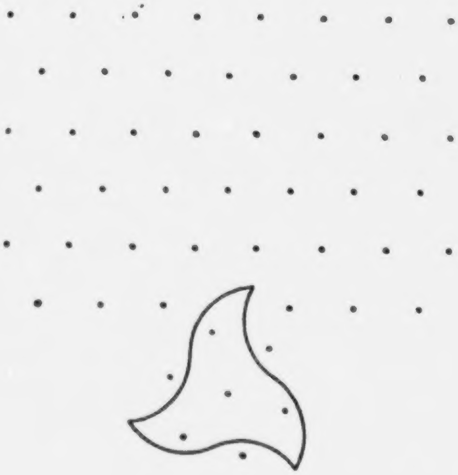
From each town eight roads should lead out:

From each town, respectively, 1, 1, 2, 4, 4, 5 roads should lead out:

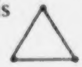


15. A. Continue drawing the pattern:

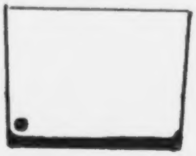




B. Measure the area on the right, if the unit area is



16. *Mitosis* (division of cells).
In this glass there is a cell.
Cells divide into two in each minute.
The first division happens at the end of the first minute. Cells are of equal size.



a. In how many minutes will there be 64 cells in the glass?
b. In how many minutes will there be 128 cells in the glass?
c. In how many minutes will there be 256 cells in the glass?
d. In how many minutes will there be about 1000 cells in the glass?
e. In how many minutes will there be 10^6 cells in the glass?
f. In how many minutes will there be 10^9 cells in the glass?

Draw a graph showing the growth of the number of cells on a graph paper. Then answer the following questions:

(i) In one hour, exactly, the glass will be full. When is the glass half full? When is it one-quarter full, and when is it 1/3 full?

(ii) If the volume of a cell were 1mm^3, about how large a glass would be filled up in 30 minutes?

[Introduction to the concept of logarithm, without speaking about it.]

17.

A. How many straight lines may be drawn through the four vertices of a tetrahedron such that each should go through at least two vertices?

B. How many straight lines may be drawn through the eight vertices of a cube such that each should contain at least two vertices?

[If your answer is 12, think again! The correct answer is 28.]

C. How many ways can you write up 5 as the sum of three positive integers, if the order of the terms is relevant?

D. How many ways can you write up 6 as the sum of three positive integers, if the order of the terms is relevant?

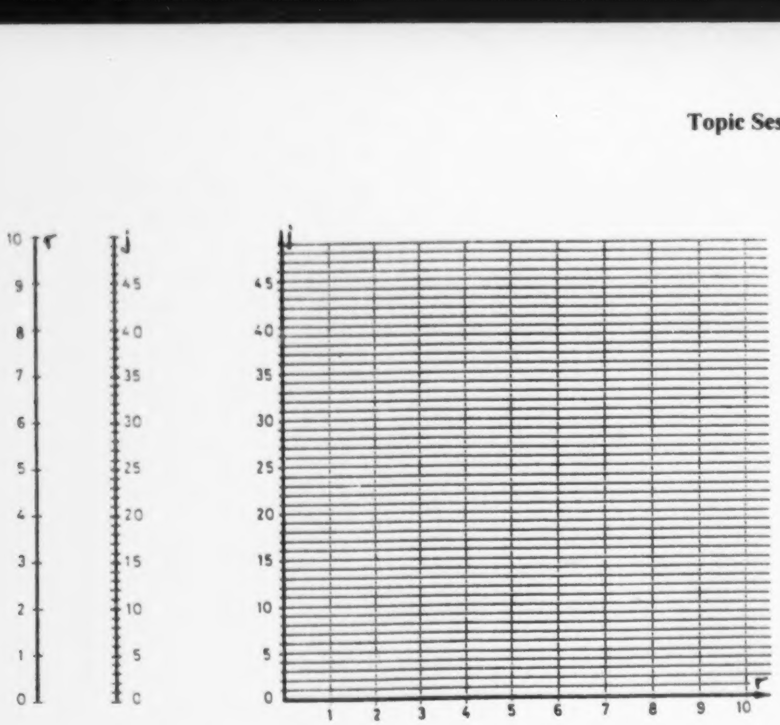
18. Which is larger? Put the proper sign of inequality, < or >, between the numbers.

2^{12} 4^{8}	3^{12} 9^{8}	2^{15} 8^{6}	6^{8} 3^{16}	8^{8} 3^{16}
4^{5} 10^{3}	2^{10} 10^{3}	2^{20} 10^{6}	4^{10} 10^{6}	5^{9} 125^{3}

19. In a chess tournament there are r participants. In case of a round robin, how many matches will be played?
Fill the blank.

The number of participants in a round-robin (r)	2	3	4	5	6	7	8	9	10
number of games (j)									

Illustrate the relation between r and j on both diagrams:


r
j
10
9
8
7
6
5
4
3
2
1
0
45
40
35
30
25
20
15
10
5
0
1
2
3
4
5
6
7
8
9
10

Which equation(s) describe(s) the relation correctly?

$\boxed{j = 4r - 10}$ $\boxed{r^2 = r + 2j}$ $\boxed{j = \frac{r^2 - r}{2}}$ $\boxed{j = \frac{r^2}{2} - \frac{r}{2}}$

$\boxed{j = \frac{r(r+1)}{2}}$ $\boxed{j = \frac{r(r-1)}{2}}$ $\boxed{j = 1+2+3+\ldots+(r-1)}$

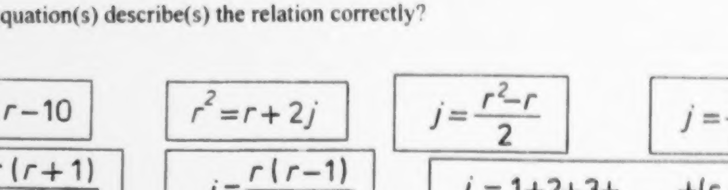
$\boxed{\frac{j}{2} = \frac{r}{2} \cdot \frac{r-1}{2}}$ $\boxed{j = 1+2+3+\ldots+r}$ $\boxed{j + r = 1+2+3+\ldots+r}$

[In answering this question the student will realize that he is able to discover the *general* pattern between r and j: this is creative thinking! This may make the young enjoy mathematics.]

20. a. How much is 1/8 of 4/6?
Briefly: (4/6).(1/8) =

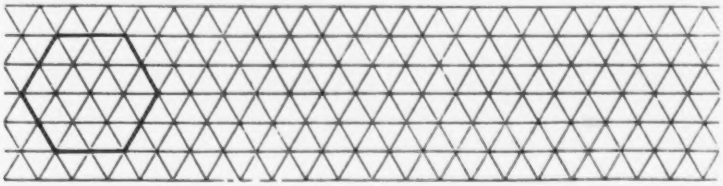
b. How much is 3/8 of 4/6?
Briefly: (4/6).(3/8) =

Show your result on the diagram:


This is the unit cell.

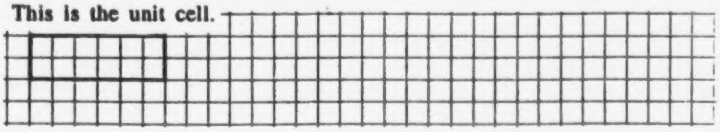
c. How much is 1/5 of 5/6?
Briefly:
Show your result on the diagram:

d. How much is 3/5 of 5/6?
Briefly:



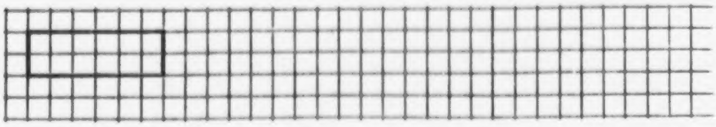
21. a. What part of 2/6 is 1/6?
Briefly: (2/6).   = 1/6
Show your result on the diagram:

b. What part of 2/6 is of 5/6?
Briefly: (2/6).   = 5/6



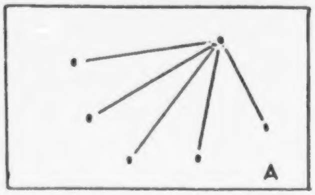
c. What part of 4/6 is 1/6?
Briefly:
Show your result on the diagram:

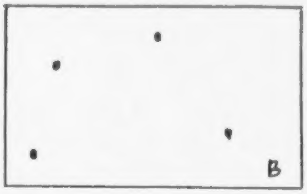
d. What part of 4/6 is 1/2?
Briefly:

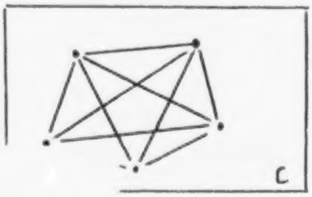


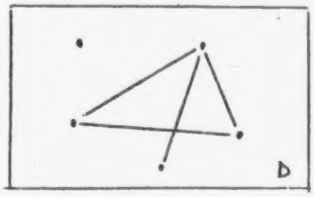
[Notice the gradual introduction to fraction multiplications.]

22. Below are diagrams about four groups of people: persons are represented by dots and connecting lines show they know each other.







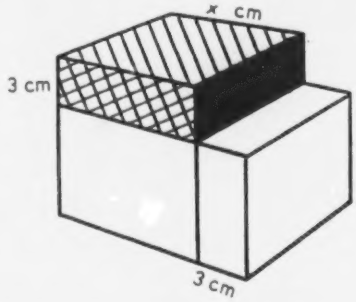


Write beside each statement the letter(s) of the group(s) for which the statement is true.
Everybody knows everybody. ....
There is one who knows evberybody. ....
Everybody knows somebody. ....
There is one who knows somebody. :.....................
There is none who knows everybody. ....
There is none who knows nobody. ....
There is none who does not know everybody. ....
It is true that nobody knows anybody. ....
It is not true that nobody knows everybody. ....

Now connect those statements which have the same meaning.

[To answer this underlined problem the basic ideas of Boolean algebra needs to be understood. If we want our children to learn these concepts at an early age, in my opinion we need to change our teacher education at the intermediate level.]

23. A cube is distorted such that its height is decreased by 3 cm, the two edges of its base are increased by 3 cm, while the other two edges remain unchanged as illustrated on the diagram:



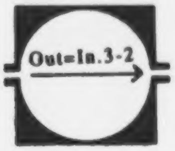
Will
a. its surface area;
b. its volume;
c. the sum of the lenghts of its edges be smaller, largert, or unchanged?

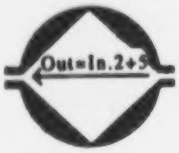
Fill in the blank for a few values of the original length of the edge of the cube (x).

The edge of the cube (cm.)							
The surface area of the cube (cm^2)							
The volume of the cube (cm^3)							
The sum of length of its edges (cm.)							
Surface area of the new formation (cm^2)							
Volume of the new formation (cm^3)							
The sum of the length of edges of the new formation (cm.)							

[This problem helps students understand that surface area and volume change in opposite directions. That among such polyhedrons with the same surface area the cube has maximum volume, and among such polyhedrons with the same volume the cube has the minimum surface area.]

24. We have two machines. One of them subtracts 2 from three times the inputed (in) number. The other adds 5 to twice the inputed number.

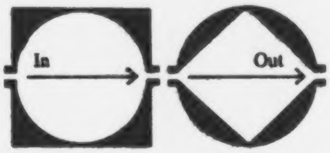




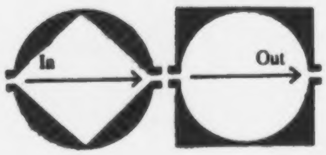
Now connect the two machines in this way:

And the other way around:

①



②



a. Which of the following open sentences describe(s) the operation of the first, and the second, connected machine? Write beside each open sentence the respective machine's number. Cross out those open sentences which belong to neither of the machines.

Out = In·3 - 2 + In·2 + 5

Out = 3·(In·2 + 5) - 2

Out = 2·(In·3 - 2) + 5

Out = In + 6

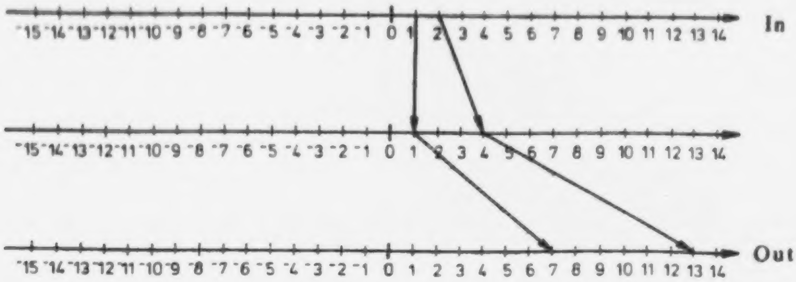
Out = In·6 + 1

Out = In·6 + 3

Out = In·6 + 13

Out = In·6 + 9

b. Illustrate the rule of machine 1 by a red pencil, the rule of machine 2 by a blue pencil:


⁻15 ⁻14 ⁻13 ⁻12 ⁻11 ⁻10 ⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
In
⁻15 ⁻14 ⁻13 ⁻12 ⁻11 ⁻10 ⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
⁻15 ⁻14 ⁻13 ⁻12 ⁻11 ⁻10 ⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
Out

[These questions are about composite functions.]

25. The following question is from the covers of the regular Hungarian mathematics textbooks, for grades 5-8:

M	A	T	H	E
A	T	H	E	M
T	H	E	M	A
H	E	M	A	T
E	M	A	T	I
M	A	T	I	C
A	T	I	C	S

In how many different ways can
you read
MATHEMATICS
going always either
downward,
or to the right?

[It is a truncated Pascal triangle. How its coefficients are related to each other can be understood by a grade six student without too much difficulty.]

The following question was published in *The Globe and Mail* (February 1, 1993), quoting *The Economist* may be interesting, of course, for high school students in Ontarion Academic Credit (AOC) programs, and their teachers:

> Given a regular pyramid with a square base, there is a ball with its centre on the bottom of the pyramid and tangent to all edges. If each edge of the pyramid base is of length a, find the following quantities: (1) the height of the pyramid; (2) the volume of the portion common to the ball and the pyramid.
>
> This problem, from an entrance paper to Tokyo University, is an example of the short of thing that makes North American educators cringe. How many North American high-school students, applying to college for math. courses, they wonder, would be able to answer it? They are asking the wrong question. The problem comes from an entrance paper for humanities students.

EPILOQUE

After reviewing the above selected problems (most of them from a grade six Hungarian exercise book), I think you would agree with me that we have room to improve our education. We do face a crisis. During this Conference there was a section on the present *crisis in curriculum*. In my opinion even the title of the section was misleading. The crisis we face, *being systemic*, means that little could be achieved with changes in the curriculum only.

There are two widely different economic systems: the planned (controlled) one, and the market system. Where there is no market system with its coordinating and incentive mechanism there should be control. In Ontario, and in Canada public schools represent a monopoly with a single philosophy of education. It has been long understood in economics that in case of monopoly, regulation and control may be necessary. In education there would be a need for a strong ministry that knows what it is doing. It would need to establish a high minimum curriculum, and to enforce teaching, learning, and teacher-educational standards. On the other hand while breaking up monopolies and fostering competitive forces has been successful in many areas of the economy with respect to improving efficiency, it has hardly been tried in Canadian education. Such an alternative would be a more market oriented system with (partially or fully) publicly funded charter schools. There are several examples that indicate that such a system would be more cost efficient, and more conducive to learning especially with regards to the poor [3], [7], [8]. *At present we have a vacuum: there is no, or only little, control in education, and little or no competition either.* The result is a wasteful, inefficient system. Also, since it is the poor and the middle class who, at present, could not afford alternative (more academic) schools, the present system is conducive to maintaining the privilege of the rich.

It is quite obvious that each country has an elite. It is not irrelevant how that elite is educated. In our system, a healthy social dynamic that would offer the poor a way via education to become a part of the leading elite does not exist. With limited resources the equality of condition can only be a dream. The *equality of opportunity* requires the responsible use of such opportunity, a simple fact our leaders, as it seems, do not keep in mind. We do not give a real chance to our brightest to become truly excellent.

Today, *the major educational choices are not about what pedagogical concepts are more efficient, but rather about what philosophy of education is best, keeping in mind the socio-political processes it may create* [9], [10]. In our system of education the most substantial decisions concerning individual advancement are almost completely left to self-motivation. Individualism is absolute, the public interest is at most secondary [9]. Teacher education is inadequate. While it is widely recognized that the semestered system in Ontario is simply bad (in my opinion an abomination), it is maintained without any substantial resistance. The system allows free choice to the uninitiated, who in the absence of proper information cannot choose well. As a result education becomes chaotic in terms of quality and the system produces a great mass of functional illiterates [9]. "Each generation of Americans has outstripped its parents in education, in literacy, and in economic attainment. For the first time in history of our country, the educational skills of one generation will not surpass, will not equal, will not even approach, those of their parents." Though this was said by Paul Copperman about the U.S. state of education more than 15 years ago as quoted in [2], I think it is equally valid for Canada today.

In 1983 The Globe and Mail wrote [4]: "[I]f we are to stem any rising tide of mediocrity in Canada, a much more determined effort ... will be required." So then what are the reasons why the system has changed so little? Clearly, we teachers, and our organizations, are partially responsible for the status quo.

Finally, allow me to quote Gábor Szegő (of Stanford University) who wrote in 1961:

> We should not forget that the solution of any worth-while problem very rarely comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be willing to make this supreme effort? The explanation is probably the instinctive preference for certain values, that is, the attitude which rates intellectual effort and spiritual achievement higher than material advantage. Such a valuation can only be the result of a long cultural development of environment and public spirit which is difficult to accelerate by governmental aid or even by more intensive training in mathematics. The most effective means may consist of transmitting to the young mind the beauty of intellectual work and the feeling of satisfaction following a great and successful mental effort.

REFERENCES

[1] Erzsébet Csahóczi, Mrs. I. Halmos, Julianna Szendrei, and Katalin Varga. (1979). *Mathematical Worksheets for Grade 6*, 6th ed. Prepared with the cooperation with the Hungarian Institute of Pedagogy (refereed), (in Hungarian).

[2] *A Nation at Risk* (1983) Report by the (U.S.A.) National Commission on Excellence, April.

[3] Milton and Rose Friedman (1984). *Tyranny of the Status Quo*, Avon.

[4] Lower education. *(1983). The Globe and Mail*, editorial, May 16.

[5] To your books. (1995). *The Economist* p. 81. (Homework helps!)

[6] One standard for all pupils? (1997). *The Economist*, September 20, pp. 27-28. (Testing national school standards.)

[7] Don't mention the 'v'-word. (1997). *The Economist*, November 27, pp. 17-18. (The case for school vouchers.)

[8] Lessons Cleveland can teach. (1997). *The Economist*, November 27, pp. 27-28. (How school vouchers work.)

[9] Values in the Public School System. (1998). Fraser Forum, January, pp. 29-30. (How foreign experts view our educational philosophy, and system. Comparison with Hungary.)

[10] New skool rules, OK? (1998). *The Economist*, February 7, 1998, pp. 57-58. (On the "progressive" school versus the "new orthodoxy".)

Group Session 5

MATHEMATICS TEACHERS' NEEDS IN DYNAMIC GEOMETRIC COMPUTER ENVIRONMENTS: IN SEARCH OF CONTROL

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ABSTRACT

The study sought to understand the needs of experienced teachers who, for the first time, are teaching geometry in a computer-based exploratory environment rather than in the traditional environment of textbook, straight-edge and ruler. Insights into these needs were obtained through a qualitative case-study, in which data was collected by observation, as well as from interviews with teachers and students and from participant journal entries. Analysis of the data showed that the four teachers participating in the study experienced an initial loss of control due to the new environment, in three categories: (1) Management control (they believed the new environment impaired their ability to maintain discipline), (2) Personal control (they were unable to determine their own expectations of the students and to assess students' achievement), and (3) Professional control (they felt they no longer had all the answers). As the teachers learned to use the new tools, however, they gained confidence in their ability to teach effectively with the new methods, and were even moved to reflect upon their previous teaching practices. Despite the apparent lack of discipline, the absence of specific expectations, and the changes in their professional role, they came to recognize and accept that in the new exploratory environment the students were learning geometry and enjoying it.

INTRODUCTION

There are many reasons why teachers teach the way they do. These include teachers' lack of knowledge about the mathematics content and lack of confidence in the "new" teaching methods. They teach using techniques that are similar to those used when they were learning mathematics.

The role of a mathematics teacher is to help students learn mathematics. Teachers have to create situations where students are 'psychologically safe', mathematical ideas are discussed, and students can work and think mathematically. Students will construct their own knowledge regardless of what the teacher does within the classroom. However, the teacher can influence which mathematical concepts are investigated.

Teachers should listen to what students say about mathematical concepts. They should encourage discussion about mathematics in their classrooms. The teacher's role should be seen as one of facilitating the development of mathematical meaning within the mathematics classroom. This role may involve changing the way many teachers view mathematics teaching and learning. However, change is difficult. Change is similar to experiencing a loss. Something that you believe in is no longer true, available, acceptable, or considered good educational practice. This loss affects people differently. How one deals with the loss determines, in some part, the success of the innovation or change.

An assumption for many new educational programs has been the premise that teachers will adapt to change and that we need only to instruct the teachers on the nature of the new change: be they curriculum, teaching techniques or assessment methods. Unlike other curriculum changes in mathematics, this change does not come from within the mathematics community as a consequence of certain cultural developments of the discipline, but as the consequence of the great changes in the social and economic reality provoked by the impact of new information technologies (Bottino and Furinghetti, 1994). Research on the reactions of teachers facing curricula innovations by which teachers reorganize their pedagogical practice and beliefs is still in its infancy (Boufi, 1994; Bottino and Furinghetti, 1994). However, in order to bring desirable changes to the system, we need to find out what is actually happening when teachers undertake changes in their teaching practice.

Romberg (1985) has pointed out that the job of teaching is to "assign lessons to a class of students, start and stop lessons according to some schedule, explain the rules and procedures of each lesson, judge the actions of students during the lesson, and maintain order and control throughout" (p. 5). Romberg believes that the mathematics curriculum is something that needs to be covered, and that few teachers see student learning and understanding as the primary goal of mathematics education.

Management of the learning environment becomes an important issue for teachers trying to make changes in their teaching practice. The teaching role has been seen as one where the teacher controls the learning environment. That control can be restrictive: directing, ordering, telling, and demanding. How teachers use this control within their classroom will clearly influence the learning environment.

The management of the learning environment must allow for students to construct their own knowledge and to take responsibility for their own learning. Students also need the freedom to discover, through exploration, different ways to build solutions. They need to spend time working with problems and searching for solutions. This process may be organized and recorded according to the predetermined plan of the teacher whose role is to facilitate the student's exploration (Burns, 1992). As such, it is important that teachers provide students with the opportunity to explore, analyze, and demonstrate their skills.

The use of the computer has been heralded as one teaching tool suitable to mathematics teachers to encourage exploration of mathematics. The expanded use of computers in mathematics education may create a shifting of roles for teachers. Assuming that teachers are willing to utilize computers in their classrooms (NCTM, 1989, p. 67) to encourage the students to explore mathematical concepts, there is a need to investigate how this utilization can be implemented. Even though the NCTM Standards (1991) have provided the impetus to change the curriculum, teaching, professional development methods, and assessment practices, there are many more factors to consider when teachers and curricula change.

There are many societal influences that affect teachers and their teaching practice. The integration of the computer into our society, the beliefs and images of teachers, and how teachers change influence mathematics teaching practices. Computer technology is visible in almost every facet of our society. Its entry into schools, however, has been slow and this is especially true for mathematics classrooms (Kilpatrick and Davis, 1993). Many reasons exist for this lack of integration into the mathematics curriculum: teachers' view of knowledge acquisition (Hannafin & Freeman, 1995), lack of availability of computer hardware (Becker, 1990), teacher anxiety towards computers (Rosen & Weil, 1995; Berebitsky, 1985), and teacher control in classrooms (Cohen, 1989; Schoefeld & Verban, 1988). In fact, the interaction between computers, teachers and mathematics is complex (Noss, 1991).

Building on recent research on teacher change, this study investigated the use and implementation of geometric construction software in mathematics classrooms. This study examined the reactions of teachers as their students explored geometric constructions in a *probing for understanding* milieu.

CONTEXT AND METHODOLOGY OF THE RESEARCH

The setting for the study was four independent schools in a Canadian city. All participants were teaching Grade 8 students (age 13-14) and were selected in the following way: a male teacher in a boys-only school, a female teacher in a girls-only school and two other teachers (one male and one female) from two co-ed schools.

A case study approach was used. The data was collected through three primary sources: classroom observations, interviews and journal entries by participants. More specifically, the data was gleaned from my field notes, transcripts of interviews with teachers, students and Head of School, a questionnaire, transcripts of classroom conversations between students and teachers, and participant's journals.

Each teacher was asked to teach the geometric construction unit, normally taught using compass and straight-edge tools, using the *Geometer's Sketchpad* computer program. I spent approximately three weeks with each teacher, observing their interaction with students and the computer software.

All interviews and classroom visits were audiotaped and I made field notes of my observations. The teacher was asked to keep a daily journal to record his feelings, concerns, successes, failures and other teaching and learning experiences. Data collection was ongoing throughout the study.

The teachers were interviewed at least four times during the study: twice before the first classroom session, at the midpoint of the in-class sessions, and at the conclusion of the classroom visits. A follow-up interview was held when the transcripts were delivered to the teachers.

A questionnaire on teacher beliefs and attitudes was given to the teacher before the class sessions to provide me with additional information about the teacher. Additional questions were asked during interview sessions based on the responses to the questionnaire and on my field notes.

Three students in each school were questioned about their interest, attitude and feelings in the areas of mathematics courses, content of this course, computers, teachers of mathematics, geometry, the software, their freedom to explore, and their anxiety about mathematics.

Each Head of School was interviewed. Questions focused on the types of supports teachers receive in the school in the areas of computer hardware, computer training, teacher professional development opportunities, and teacher change.

ANALYSIS OF THE CASE STUDIES

Case Study 1

The 1995-96 school year was Cathy's seventeenth year of teaching. She works in an all-girls independent school in a large metropolitan city in Canada. Cathy had experience teaching in a government school in the Bahamas and eleven years at another all-girls, independent school in this same city before beginning to teach at St. Francis School six years ago.

The case of Cathy demonstrates that the need for control over the teaching environment is based on a personal philosophy towards instruction. Cathy's control over her environment required her to be organized and to develop her own classroom activities. Cathy permitted her students to explore the software and share in the formulation of definitions and the discovery of relationships.

	Cathy	Karen	Simon	Mike
No. of Years Teaching Math	11	3	3	14
Math Experience	extensive	little	extensive	extensive
Computer Experience	extensive	limited	extensive	medium
Math Exploration	extensive	none	none	extensive
Transition to Taken-as-Shared	integrated	beginning transition	beginning transition	progressing
Perceived Role of the Teacher - Pre-study	Facilitator	Transmitter of Knowledge	Provider of Information	Provider of Information
Perceived Role of the Teacher - Post-study	Facilitator	Transmitter of Knowledge	Facilitator	Student-focused Guide
Control Needs - Pre-study	low	high	moderate	low
Control Needs - Post-study	low	moderate	moderate	low
Control Techniques	designing worksheets	increased own proficiency	evaluation of students	designing worksheets
Control of Technology	comfortable	nervous / concerned	comfortable	comfortable

Table 1: A Cross-Case Summary of the Four Teachers

Cathy is a reflective practitioner who, through written and mental practice, makes adjustments to her pedagogical techniques before, during and after each lesson.

Certain characteristics of Cathy's efforts to implement the geometry curriculum unit were noteworthy:

- Cathy is a reflective practitioner.
- Cathy is an organized teacher.
- Cathy felt comfortable in a "taken-as-shared meaning" (Wood, Cobb and Yackel, 1991) classroom setting.
- Cathy's loss of classroom control was temporary and quickly resolved.
- Cathy believes that student's becoming active independent learners is quite important.

Case Study 2

The 1995-96 school year was Karen's eleventh year of teaching. Karen works in a Grade K-12 independent school in a large metropolitan city. The school is a co-educational environment, attracting students from various areas of the city.

The case of Karen demonstrates that the perceived need for control over the teaching environment is an important issue in teacher change. Karen's perception of the role as a teacher created a tension between having a structured classroom where the meanings are teacher-directed and a flexible classroom where taken-as-shared meanings are formulated.

The variation of the teacher's need for and actual control over their teaching environment emerged from the data. Karen's transition from a traditional, structured and controlled environment to lack of control in the new environment and, finally, to maintaining a new type of control was a significant finding.

As I studied Karen's efforts to implement the geometry unit, several characteristics began to emerge:

- Karen was a structured teacher.
- Karen found it difficult to reflect, in written form, on her teaching practice.
- Karen was beginning to show some movement from teacher-imposed meanings towards a "taken-as-shared meanings" (Wood, Cobb and Yackel, 1991) with her students.
- Karen's perceived role of the teacher had a major influence on her instructional techniques.
- The degree of control Karen felt she needed in her teaching practice created a tension that was not resolved.
- Karen felt a lack of control over her use of the technology.

Case Study 3

The 1995-96 school year was Simon's third year of teaching. The school is located in a suburb of a large metropolitan city, surrounded by open fields on one side and a housing development on the other.

At the start of this study, Simon viewed the teacher's role of teaching as a transmitter of information. After his work with the software, he saw a need for the teacher to become a facilitator whose job was to guide the students.

My work with Simon revealed a number of characteristics:

- Simon is becoming a reflective teacher.
- Simon is a structured teacher.
- Simon is aware that, in the process becoming an advocate of using taken-as-shared meanings with his students, he was often very tense in the classroom from a more traditional approach.
- Simon uses evaluation of students to maintain control over his classroom environment.

Case Study 4

The 1995-96 school year was Mike's eighteenth year of teaching and his fifth year at Stevenson College. Mike teaches in a boys-only independent school in a large metropolitan city in Canada. Prior to moving to Stevenson College, he taught thirteen years at an all-girls independent school in the same city. Mike changed his view of the role of the teacher from that of a "provider of information" to that of a guide. He believes that Middle School students should have the same opportunities to explore and share experiences in mathematics as those currently afforded his senior students.

Mike had allowed them to take control of their own learning. This new class of 'explorers' and 'experts' became the focus of Mike's use of the computer software.

Throughout my study, certain characteristics of Mike's teaching practice were noteworthy:

- Mike has a flexible approach to classroom management.
- Mike is an organized teacher.
- Mike allows the students to participate in taken-as-shared meanings within his classroom.
- Mike's perception of the role of the teacher allowed for students to take a more active role in their own learning.
- There is a strong sense of exploration and discovery in Mike's teaching practice.

FINDINGS

Teacher Control

The issue of control permeated the entire study. What were the teachers controlling? Each teacher had a number of objectives for their lessons. The teachers wanted their students to learn the geometric relationships and they wanted to be able to evaluate how well their students grasped the concepts. Jaworski (1994) recognized the issue of control in her research. In the present study, how the teacher reacted to this control issue contributed to their perceived degree of success of the program.

Teaching strategies are closely linked to classroom management strategies (Keller, 1996). Teachers in Keller's study noted that students are more on task and self-managed in a computer classroom than in the regular mathematics classroom. Teachers also noticed that there was an increased noise level in the computer lab. These changes may not be directly linked to the use of the computer but as a result of other changes that were made to accommodate their use. Keller's teachers took more interest in their students' successes and allowed the students the freedom to explore learning materials. These changes were enacted by teachers 'letting go of the reins'.

All four teachers in the present study faced the issue of control. Karen, Simon, and Mike made explicit reference to their temporary loss of control while Cathy made implicit reference to this issue. Of the four teachers, Karen's experience was most significant and will be used to develop a new understanding of the issue of teacher control.

Karen perceived a significant loss of control early in the study. She felt uncomfortable with the software and felt that she was not in control of the learning environment. Karen's students, although they were conditioned to ask and expect Karen to answer their questions, recognized this loss of control. Her students began to take responsibility for their own learning. They began to share conjectures and to help each other develop the skills necessary to explore geometric relationships using the software. Karen regained her sense of control by the third day of the study through her successful interaction with the software the previous evening. By this time, however, her students were feeling comfortable with the software and helping each other. By the sixth day, Karen regained her confidence in her ability to teach geometric constructions using the software. Both Karen and her students were transformed by the experience. The students felt in control of their own learning and Karen regained her sense of control over the learning environment and her confidence as a teacher.

Karen's experience is similar to the experiences Frobisher (1994) noticed happening to teachers when a problem-centred classroom diverges from the traditional model. The sense of insecurity that teachers experienced using the computer software is consistent with the experiences of the teachers in a problem-based mathematics classroom.

The issue of teacher control can be viewed from three perspectives: management control of the learning environment, personal control, and professional control.

Management Control

The most significant effect on the teaching pedagogy of the teachers involved in this study relates to their interaction with their students. The teachers were, to various degrees, inculturated into the traditional teaching paradigm where the teacher structures the classroom so that the teachers can be the authority. Berebitsky (1985) found that elementary school mathematics teachers have a low level of mathematical background. There are a number of problems inherent in this situation. Teachers are not confident in their mathematical ability and, therefore, the textbooks are taken as the authority for mathematics. Steffe (1990) suggests that the mathematical concepts and how they are taught seldom get questioned.

Teachers having low level of mathematics background or those who depend on the textbook as if it were the curriculum, tend not to respond favourably to suggestions that they teach in an exploratory mode. They perceive this environment as being too difficult of control and that it requires the teacher to tolerate uncertainty about what the students are learning (Schoefeld & Verban, 1988; Cohen, 1989).

The role of the teacher in mathematics education also influences the control mechanisms the teacher places on the classroom environment. When teachers chose various situations for their classroom, they make judgments about the relevance of the situation to their students and how likely the students are to "bump" into the appropriate mathematics in the course of investigating the problem (Lappan & Briars, 1992). These activities will vary depending on the level of control and the tolerance level a teacher has within the classrooms. These levels of control and tolerance levels may restrict the use of cooperative/collaborative learning activities used by the teacher. Johnston, Johnston and Stanne (1986) found that students working in cooperative learning groups had increased achievement within computer-based environments. Even with these findings, teachers may choose to have students work one to one with a computer simply to minimize the noise. In so doing, the teacher may inadvertently lessen the opportunities for students' discourse and shared meanings within a small mathematical community.

Personal Control

Teaching can be an isolated activity. The teacher is expected to teach around 30 students, maintain control, and inspire the class to learn (Cuban, 1986). Compound this problem by introducing a computer software tool and tensions develop between the teacher's perception of their role as a facilitator within the classroom and their personal control needs for perceived control of the learning environment.

These personal control needs are expressed in many forms. The need for mutual trust within the learning environment between the students and the teacher, the need of being the authority within the mathematics classroom and the ability to freely admit mistakes are within this category. How a teacher perceives herself within the classroom and how the teacher reacts to personal rather than professional change has an impact on the degree to which change is accepted by the teacher.

Karen provides an interesting backdrop for the importance of personal control in an elementary school mathematics teacher's practice. While she had some concerns over the content of the geometry program, she was most concerned with her personal control over her environment. She wanted to maintain a personal presence in the classroom and expects respect from her students. She felt that this respect was synonymous with her personal control within the classroom. Karen made some changes in her teaching practice. These changes, however, were closely related to her feeling comfortable with and in control of her personal acceptance of the need to share the authority of mathematics. She also felt comfortable to share her lack of complete understanding of the software with her students. It was at this point that she began to recognize her control over the teaching environment increase.

Simon, as a new teacher, had similar hopes in the classroom. He wanted students to recognize him as an individual. He was not concerned about making mistakes but he was unable to freely inform his class that mistakes were part of life. He realized that he didn't have to know everything and that sharing knowledge with the students actually allowed his personal control to increase.

Mike and Cathy appear to be very comfortable with their role both in and out the classroom. They have a personal interest in investigating mathematics and freely admit to their students that they make mistakes. They do not need to be the centre of attention and, perhaps as they have both taught for a number of years, felt confident in their abilities to make changes within their teaching practice without creating a loss of personal control over their environment.

The issue of personal control is important to new teachers to the profession and those new to teaching mathematics. The need to be the centre of attention and to be the mathematics authority in the classroom does influence how a teacher reacts to change within their classroom. Students benefit from seeing teachers as evolving, learning members of the mathematics community. Rather than providing students with information and then determining if they have captured the concepts, knowledge and skills, teachers will need to become a part of a learning community and act as a model and a participant.

Professional Control

Prospective teachers enter a profession steeped in tradition and history. As a profession, teachers are well regarded in some communities and not in others and may experience some trepidation about their role within the community. Within the independent school system, teachers are usually well regarded for their hard work and dedication to the profession.

All four of the teachers in this study agree that there are many roles for the teacher within the classroom. They agree that being good in mathematics is important but not essential. The ability to motivate students is a key factor, according to Simon, while Cathy believes that teachers should ask questions to encourage students to explore mathematics. Both Cathy and Simon, by the end of the study, saw the teacher's role as that of a facilitator while Mike used the word guide to describe his role in a more student-focused classroom environment. Karen continued to believe that her role was to 'teach'. That is, she should provide an environment where she is the transmitter of knowledge to the students. In each case, the perceived role of the teacher dictated the types of questions posed, the distribution of the worksheets, and the interaction between teachers and students.

How a teacher perceives the role of the teacher will contribute to the type and degree of control used in the classroom. A teacher who believes that the teacher should be a facilitator will naturally maintain a different form of control over the classroom. A facilitator will have less difficulty with open-ended activities and will invite questions from the class that will be different in scope and depth than from a teacher who believes that students need to be told what to learn and under what conditions. The transmission-type teacher will be less likely to open the students to new questions and interaction, the building blocks of a mathematical community.

CONCLUSIONS

Teachers play a central role in establishing the mathematical quality of the learning environment for students and in establishing norms for mathematical aspects of students' activity (Yackel and Cobb, 1996). This implies that the teacher does not take a passive role in the constructive perspective but plays a critical role as a representative of the mathematical community. Given this central role, what influences come to bear on the role of the teacher in mathematics education?

The teacher's view of learning about mathematics and mathematics teaching clearly affects how teachers present the course material. The teacher needs to do more than just change the nature of the classroom task from teacher-directed to student-directed. Social constructivism implies that students need to communicate with each other. This communication could cause anxiety for teachers who feel that classrooms should be quiet, or that only one person should be talking at a time.

The use of the word 'control' conjures up different images for each of us. It is an emotional word that can be used negatively to suggest that the teacher is not giving students any freedom to develop their own thoughts (Jaworski, 1994). It can also mean that the students take responsibility for their own learning. Classroom control is important for teachers and is used to influence the way students think and behave within the classroom. Teachers use of their inherent control within the classroom will influence the type and form of activities that take place within the classroom. This control can be used to limit interaction between student by reducing the noise level to a minimum or nil and by insisting on individual work. However, as Jaworski found in one of her case studies, control can also create an environment in which mathematics thinking is fostered.

We need to develop a careful understanding of the settings that encourage teachers to learn to use these new teaching environments and materials. We need to determine the real costs of teachers learning to teach geometry. We also need to empower teachers to create an experimenting environment in their classrooms. Teachers need to be observed in computer exploratory environments so that we can determine their learning needs so they can provide this educational experience for their students.

Teachers experienced an initial loss of control in this environment. As the teachers gained confidence in their own use of the software and recognized that students were experiencing success, teachers began to regain their sense of control. The investigation also reveals that teacher control can be expressed in one of three categories: management control of the learning environment, personal control, and professional control.

The implication for teacher education is that preservice and inservice teachers should be given a mentor or coach to reinforce the premise that, although the teacher will experience a temporary loss of control, increased confidence in mathematics and experience in other software packages, will be helpful for teachers attempting to introduce dynamic geometric software packages into their classrooms. The implication for mathematics education is that students thrive in dynamic geometric software environments when teachers maintain control over the management of learning, their own personal expectations, and their role as a professional.

Teacher education programs should include activities that place teachers in learning environments where they can explore mathematics, interact with their peers though discussion and case studies, and to work with dynamic computer environments. These dynamic computer environments provide an environment where teachers and students can interact and share their conjectures and findings with each other. Educators of teachers should provide opportunities within their curriculum for teacher exploration using these computer-based tools. Moreover, research could be conducted to develop a better understanding of how the inservice and pre-service programs provide potential and current teachers with insight on creating a mathematics learning community within their classrooms.

Suggestions to Teachers in Making the Shift in Control

From this study, a number of suggestions can be made to teachers, consultants, teacher educators, and other researchers which may assist them in implementing a geometric exploratory classroom using a dynamic geometric software program. In point-form, the following items would be most helpful and necessary for teachers attempting to make this shift:

- the availability of a mentor. Having a coach available for technical and mathematical content support would be helpful.
- that sufficient time be provided for teachers embarking on this process.
- a valuing of the process by the administration of the school, the district officials, and the teachers themselves. In this study, the department Heads and Heads of the school were very supportive of the project.
- reflection on the activity through journals and meetings with other teachers.
- a willingness to focus on the shift in control. These teachers participated freely and willingly in this project. What would have happened if every teacher had to be involved in the program as is normally the case in curriculum reform?
- specifically, teachers need to consider their noise tolerance level, class structure and the role of cooperative/collaborative activities in the mathematics classroom, the shift in control to the students, and ways still to feel 'in control' as a facilitator as well as how effective student evaluation will be handled.

SUMMARY

Professional control is important in middle school mathematics teaching. Teacher educators can assist teachers to maintain a level of control over their professional lives by providing them with the tools to be mathematical explorers. Teachers need to be placed in learning environments where they can explore mathematics, interact with their peers though discussion and case studies, and work with dynamic computer environments. These dynamic computer environments provide an environment where teachers and students can interact and share their conjectures and findings with each other. Teacher educators should provide opportunities within their curriculum for teacher exploration in these computer-based tools.

REFERENCES

Becker, H. (1990). *Effects of computer use on mathematics achievement*. Baltimore, MD: Center for Research on Elementary and Middle Schools.

Berebitsky, R. D. (1985). *An annotated bibliography of the literature dealing with mathematics anxiety*. Indiana University (ERIC ED257 684).

Bottino, R. M. and Furinghetti, F. (1994). Teaching Mathematics and Using Computers: Links between Teacher's Beliefs in Two Different Domains. *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, Lisbon, Vol. 2, pp. 112-119.

Boufi, A. (1994). A Case Study of a Teacher's Change in Teaching Mathematics. *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, Lisbon, Vol. 2, pp. 120-127.

Burns, M. (1992). *About teaching mathematics: A K-8 resource.* White Plains, NY: Math Solutions Publications.

Cohen, D. (1989). Educational Technology and School Organization. In R.S. Nickerson and P. P. Zodhiates (Eds.), *Technology in education: Looking towards 2020* (pp. 231-264). Hillsdale, NJ: Lawrence Erlbaum.

Cuban, L. (1986). *Teachers and machines.* New York: Teachers College.

Frobisher, L. (1994). Problems, Investigations and an Investigative Approach. In A. Orton and G. Wain (Eds.), *Issues in teaching mathematics* (pp. 150-173). London: Cassell.

Hannafin, R. D. and Freeman, D. J. (1995). An Exploratory Study of Teachers' View of Knowledge Acquisition. *Educational Technology*, January-February, 29-56.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry.* London: The Falmer Press.

Johnston, R. T., Johnston, D. W. and Stanne, M. B. (1986). Comparison of computer-assisted cooperative learning, competitive and individualistic learning. *American Educational Research Journal*, 23(3), 382-392.

Keller, C. M. (1996). Networked Instructional Computers in the Elementary Classroom and Their Effects on the Learning Environment: A Qualitative Evaluation. *Journal of Research in Computers in Education*, 28(3).

Kilpatrick, J. and Davis, R. B. (1993). Computers and curriculum change in mathematics. In Keitel, C and Ruthven, K. (eds.), *Learning from Computers: Mathematics Education and Technology.* New York, Springer-Verlay, 203-221.

Lappan, G. and Briars, D. (1995). How should mathematics be taught? In I. Carl (Ed.), *Prospects for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation of standards for school mathematics.* Reston, VA: The Council.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: The Council.

Noss, R. (1991). The Social Shaping of Computing in Mathematics Education. In D. Pimm and E. Love (Eds), Teaching and learning school mathematics (pp. 205-220). Milton Keyes: Open University.

Richardson, V. (1990). Significant and worthwhile change in teaching practice. *Educational Researcher*, 19(7), 10-18.

Romberg, T. A. (1985). Research and the Job of Teaching. In T. A. Romberg (Ed.), *Using research in the professional life of mathematics education* (pp. 2-7). Madison: Wisconsin Centre for Education Research.

Rosen, L. D. and Weil, M. M. (1995). Computer Availability, Computer Experience and Technophobia Among Public School teachers. *Computers in Human Behavior*, 11(1), 9-31.

Schoefeld, A. H. and Verban, (1988). Computer Usage in the Teaching of Mathematics: Issues that Need Answers. In D. A. Grouws and R. Cooney (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 169-193). Research Agenda for Mathematics Education. Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (1990). On the Knowledge of Mathematics Teachers. *Journal for Research in Mathematics Education*, Monograph Number 4, 167-184.

Wood, T., Cobb, P. and Yackel, E. (1991). Changes in teaching mathematics: A case study. *American Educational Research Journal*, 28(3), 587-616.

Yackel, E. and Cobb, P. (1996). Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.

AD HOC SESSIONS

Ad Hoc Session 1

TEACHERS TAKING ACTION: USING THE NATIONAL MATHEMATICS PROFILE TO IMPROVE TEACHING AND LEARNING

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INTRODUCTION

The National Professional Development Program (NPDP) was an Australian federal government program aimed at improving teaching and learning in Australian schools. Over a three year period, from 1994 to 1996, $60 million funding was made available to assist state Departments of Education, teacher professional organisations, universities and other educational agencies to develop teacher professional development programs across all learning areas. Historically, the NPDP arose as a natural extension of efforts towards development of national curriculum guidelines. Development of National Statements and Profiles in eight key learning areas began in 1989; in English, mathematics, science, technology, languages other than English, health and physical education, studies of society and environment, and arts (for the mathematics documents, see Australian Education Council, 1991, 1994).

The National Statements and Profiles were intended to provide a framework for curriculum development by education systems and schools, and a foundation for courses that would meet students' needs while reflecting advances in knowledge of the particular learning areas and how students learn. In July 1993 the Australian Education Council agreed that use of the National Statements and Profiles would be the prerogative of each state and territory. Since that time most states have adopted to some degree some form of the Profiles. For example, Victoria developed modified versions of the Profiles, called *Curriculum and Standards Frameworks* (for mathematics, see Board of Studies, 1995), and are using them across the eight Key Learning Areas (KLA's). In Western Australia, a similar situation occurred, with the documents called the *Student Outcome Statements* (for mathematics, see Education Department of Western Australia, 1994). The National Profile in mathematics (Australian Education Council, 1994) and its state-developed counterparts provide a framework to conceptualise mathematics curricula with a focus on holistic, integrated teaching and assessment processes, and a developmental approach to interpretation of student learning.

Within the context of this national program, this paper reports on how primary and secondary teachers in two states developed their understandings of and capabilities with: a new curriculum document, mathematics topics, innovative teaching and assessment methods, how students learn mathematics, and how a school staff can design, monitor and revise their own professional development program. The mathematics learning area of the NPDP projects in Western Australia and Victoria are the focus of analyses and discussions.

In Western Australia the NPDP projects in mathematics were called *NPDP Maths*, while in Victoria they came under the project title *Maths in Schools*. In each state, a professional development program was established under the direction of a management or steering committee consisting of representatives from the mathematics teachers' professional organisation, the state department of education, Catholic and independent school authorities, and universities. Also, in each state, funding allowed a project officer to be hired who acted as the person responsible for the running and monitoring of the programs on a daily basis.

Schools were required to apply for participation in the NPDP, with those selected provided with funding for specified numbers of teacher relief days. Funding was also provided for the time of a 'facilitator' assigned to each school. Facilitators were generally university based mathematics educators, but the group also included other mathematics educators with experience as a mathematics professional development consultant. The model of professional development adopted in both states was that of school-based, tailor-made programs that served in most cases in an action research capacity.

METHOD

The primary principle of the *NPDP Maths* and *Maths in Schools* was that teachers be supported in working together to develop and direct their own professional development agendas. A team of teachers from a school, cluster of schools (i.e., located closely geographically), or school network (a regional grouping of schools by the state government for networked information and support mechanisms) would identify a local issue of concern in mathematics education, and then design and implement a series of professional development activities to address this issue. These activities could include any combination of workshops, discussions, classroom trialing, professional reading, resource collection or development, project documentation, or any other activities that encourage teacher collaboration or support teachers' efforts to make relevant changes in their teaching practices. A university based mathematics educator, called a facilitator, was appointed to assist each project team in whatever ways they mutually determined would be most appropriate and effective. Each project selected a key teacher who would act as the main liaison person between the project team and the facilitator or the project officer for *NPDP Maths* or *Maths in Schools*. Since the entire program was under NPDP funding, use of the Mathematics National Profile (or state equivalent) was required by each project. The nature of this use was however determined by individual project teams.

Development and Implementation of Projects

All project groups were encouraged to develop a brief action plan indicating how they would tackle their chosen issue. This plan was developed in conjunction with the university facilitator, taking into account available curriculum days (i.e., student free days) and staff meeting times, additional possible meeting times, and the time availability of the facilitator. The plan also included consideration of the teachers' needs in relation to the chosen focus, along with the areas of expertise of the facilitator. It was required that the plans incorporate opportunities for teachers to meet and share, discuss and reflect upon activities and issues related to implementation of their project plan.

The foci of the projects varied widely from school to school, as did the programs they designed to implement their plans. The project foci included: a topic strand that had been neglected in the school (eg. space or chance and data), use of technology (such as graphics calculators), the primary to secondary school transition, problem solving, integration of mathematics with other curriculum areas, improvement of teaching strategies and assessment, and reporting.

Program Evaluations

In Western Australia, formal evaluation of the projects was conducted across all NPDP learning areas by external researchers. *NPDP Maths* also conducted its own evaluations through discussions with participants, observations made by facilitators, reports from members of the management committee, project summaries and reports prepared by schools, and written feedback forms completed by participating teachers, school administrators and facilitators. In Victoria, all formal evaluation was conducted by the project officer for *Maths in Schools*. The data sources were similar to those used for *NPDP Maths*. The data were analysed in the context of the overall goals of the *NPDP Maths* and *Maths in Schools* programs. In addition, inductive analysis (Glaser and Strauss, 1967; Powney and Watts, 1987) was used to identify common issues or happenings that emerged across projects.

RESULTS

This discussion will provide an overview of the happenings and outcomes of the NPDP mathematics projects in Western Australia and Victoria from 1994 to 1996. It is structured around the aims of the NPDP (the nature of outcomes and the mechanisms by which they were achieved), and how the action research professional development processes affected teachers.

Use of the National Statement and Profile in Mathematics

A stipulation of being awarded NPDP funding was that a school use the National Profile (or equivalent document) as a reference for the focus and subsequent development of their project. In Western Australia the state-developed version is the mathematics *Student Outcome Statements* (SOS) (Education Department of Western Australia, 1994), while in Victoria it is the mathematics *Curriculum and Standards Framework* (CSF) (Board of Studies, 1995). The degree to which schools used these documents and the nature of use of the documents varied considerably between schools. Some schools used them extensively, as a basis for their whole project, including using them as a guide for program auditing and planning, for re-writing units or planning learning activities, or for use as a guide to assess outcomes of trialed activities. Other schools used the documents less comprehensively, for example, by using them to check school programs and activities against overall course advice and structure. The most common focus for projects across the three years was to deal with one mathematics strand in detail. For example, chance and data across the primary grades, or algebra in grades 7 and 8. In the final year of *Maths in Schools*, several projects (approximately one fourth of them) chose the use of technology as a project focus and thereby explicitly addressed the Tools and Procedures strand of the CSF. In that same year (1996) nearly one fourth more of the projects were focused on assessment or reporting. This latter focus was a strong reflection of the social climate of that time, where teachers would soon have to report to parents and other individuals by reference to the CSF.

Most projects (greater than 80%) reported as a result of participating in NPDP to have attained greater familiarity with the content, language and structure of the SOS or CSF. They indicated that participation helped them focus on SOS or CSF outcomes, while concurrently developing capabilities to use the SOS or CSF for both planning and assessment of learning activities. Many schools also reported that as a result of the program they now have a new mathematics policy or mathematics scope and sequence in place that is consistent with the SOS or CSF, or new teaching and assessment methods that concur with SOS or CSF outcomes. A number of schools developed new school-based curriculum documents that follow the SOS or CSF and are already or will soon be using these documents extensively. Some of the primary schools reported that participation in NPDP provided them with the capabilities to commence professional development in other strands of mathematics, or even in other key learning areas. A few of the teachers reported that developing competence with use of the SOS or CSF was a very time consuming process that sometimes left them feeling overwhelmed, but which simultaneously enhanced their teaching practice and their students' learning.

Renewal of Teachers' Discipline Knowledge and Teaching Skills

Teachers reported that participation in the NPDP helped them to broaden their knowledge of mathematics and mathematics teaching in relation to a number of areas, including: (i) particular topics (for example, space, algebra, or chance and data), (ii) the integration of technology into mathematics teaching (most commonly the use of graphics calculators), (iii) the use of a wider range of resources to support teaching (for example, more extensive use of manipulatives and other concrete learning materials), (iv) the use of a broader array of assessment tasks (in particular, more open-ended types of tasks), and (v) the use of more 'real' life mathematics activities that provide students with more meaningful learning experiences.

Many teachers also reported that along with improvement in their teaching practice, participation in NPDP increased their confidence in mathematics teaching. Some teachers indicated that their enhanced teaching practice impacted on students' attitudes towards mathematics, with students becoming more positive and enthusiastic towards their mathematics learning and displaying more enjoyment of it.
Most of the university facilitators indicated that part of their involvement with schools involved them in presenting workshops on various mathematics content topics (such as space), or various teaching techniques or assessment methods (such as use of task centres for problem solving experiences, or development of Rich Assessment Tasks (RAT)). A number of the facilitators were also involved in working with small groups or individual teachers in their classrooms, working on alternative teaching or assessment methods. An even greater number of facilitators indicated that their role involved them as a provider of information on the nature and potential of teaching resources not yet known to the teachers, or information from research in mathematics education that would be of relevance to the school's project focus. Thus, in these avenues of input from the university facilitator, teachers were further able to renew their discipline knowledge and teaching skills.

Enhancement of the Professional Culture of Teachers

Most schools indicated that involvement in NPDP impacted upon the professional interactions of teachers by providing a mechanism through which they worked together collegially in sharing and discussing ideas and events related to mathematics teaching. In this way, the program acted as a catalyst or focus for dialogue about a wide range of issues, both theoretical and practical. Some teachers were involved with in-servicing other teachers, while others, in particular the key teachers, served as mentors to less experienced or confident teachers. Key teachers indicated that they were able to develop their leadership skills as a consequence of taking on a leadership role. Many teachers explicitly indicated their participation in NPDP caused them to become more reflective in their teaching practice. Other teachers, although not stating explicitly that they had become more of a reflective practitioner, indicated implicitly that this was so because they commented that they had begun to think more extensively about curriculum issues and had become more questioning of their own teaching and the actual nature or extent of their students' learning.

Key teachers noted that involvement in NPDP promoted such things as: greater respect and mutual support amongst teachers, greater awareness of what other teachers were doing in their mathematics teaching, particularly those in other grade levels, greater interest in, awareness of and enthusiasm for issues related to mathematics teaching, and more willingness to talk about mathematics teaching. When asked what were the most valuable aspects of their school's project, teachers included the structure of the professional development program in that they worked as part of a team and shared and discussed things with other teachers. They felt this was a most rewarding and productive format for conducting effective professional development. In addition, they were of the opinion that an essential component of this growth was the input from a university facilitator, who served as a resource and someone with different expertise and perspectives, as well as a mentor and provider of support and encouragement.

Promotion of Partnerships Between Groups Responsible for Education

The *NPDP Maths* management committee and the *Maths in Schools* steering committee that served to oversee the mathematics projects in each state were composed of representatives from the government and Catholic school systems, representatives from each of the state's universities, and representatives from the host organisation, the mathematics teachers' professional organisation. Some of these representatives were also present at the orientation and evaluation seminars held for key teachers and facilitators. In these ways, virtually all formal authorities involved in mathematics education in Western Australia and Victoria contributed to the development and success of the NPDP projects. The contacts and networking that occurred as a result of this involvement increased awareness of each contributor's role in mathematics education in the state, while also strengthening communication links. It is anticipated that these links will be maintained in both formal and informal sharing of needs, plans and events.

Through the association of a university facilitator with each project, both school and university personnel had opportunities to develop appreciation of the nature and role of each person's position in mathematics education. University facilitators commented that they had learned much from ongoing direct contact with teachers and students. Many stressed that this learning would impact significantly upon their own work as an academic since it gave them new insights into the realities and constraints of today's classrooms and new perspectives on the relationship between theory and practice in education. It simultaneously provided examples and activities for use in their university teaching or future in-service work with teachers.

The program supported communication links between schools through the projects developed around school networks or clusters, and through provision of seminars in which teachers were able to meet and discuss project issues with teachers from other schools. In addition, communication links between schools and parents were enhanced because some schools conducted information sessions for parents to inform them of the project aims and outcomes, and other schools involved parents through use of new reporting forms.

Dissemination of Outcomes

1. Development of resource materials

The *Maths in Schools* program produced four resource manuals over the three year period of its operation that were distributed to all schools involved in the program, all facilitators and Mathematics KLA network leaders throughout the state. The *NPDP Maths* program produced a monograph containing accounts of teachers involved in using the *Student Outcome Statements*.

2. Conferences

The NPDP *Maths* organised a two day conference during the October school break during 1995 and 1996. This conference was open to all teachers in the state, and plenary and workshop sessions were offered that focused on topics and issues related to becoming familiar with and using the Mathematics *Student Outcome Statements*. In March, 1996, the *Maths in Schools* host organisation, the Mathematical Association of Victoria (MAV), participated in a Victorian Teachers Network Coordinating Committee (VTNCC) Partnership Conference. Additionally, the MAV has an annual conference each December in Melbourne that is attended by about 2000 classroom teachers and others involved in mathematics education. Many of the schools involved in *Maths in Schools* prepared workshops or presentations for these conferences. In this way, the processes and outcomes of their projects were shared with other teachers from around the state as well as some educators from other states.

3. Use of electronic communication

Use of electronic communication by *Maths in Schools* participants was encouraged, but optional, because not all schools, particularly primary schools, had easy access to the Internet. Use of electronic communication included: facilitators and project schools communicating by email, schools communicating by email, facilitators or teachers communicating with the project officer by email, and access to useful material via the Internet.

Teacher Involvement in Action Research Professional Development

The form of self-directed professional development adopted by the NPDP projects could be said to be action research in that it was a cyclical model of planning, acting and observing, and reflecting (Kemmis and McTaggert, 1988). The outcomes of components of the program are outlined below, as are the enhancement of knowledge in mathematics education, and involvement of teachers and schools in cooperative endeavours.

1. Identifying a Project Focus

An aim of the professional development model was to have teachers work collaboratively to identify an area in mathematics teaching and learning that was mutually determined to be in need of attention or improvement. By having all teachers play a role in this decision making process it was believed they would feel more ownership of the project, and it would guarantee that each individual was working on an issue of personal relevance and interest. The degree to which schools involved all staff in the initial decision as to the project focus varied widely from school to school. In many cases the decision was made mutually by the entire project team, but it was more common for a subset of that team to make the decision. There were also situations in which the decision was made by only one or two people (for example, the key teacher and the principal or department head). In most cases where not all project members were involved in the decision about the project focus, there did not appear to be strong opinions that a different focus should have been chosen. In other words, the focus chosen, though not determined mutually, was mutually an issue of concern to staff and administration.

2. Development and Implementation of an Action Plan

All project groups were encouraged to develop a brief action plan indicating how they would tackle the chosen issue. This plan was developed in conjunction with the university facilitator, taking into account available curriculum days and staff meeting times, additional possible meeting times, and the time availability of the facilitator. The plan also included consideration of the teachers' needs in relation to the chosen focus, along with the areas of expertise of the facilitator. These programs varied substantially in content, distribution of meeting or workshop times, and overall length of involvement. An example of one plan is given below:

> *Suburban primary school: Development of mathematics teaching and learning strategies for multi-age (multi-grade) classrooms*
>
> Actions:
> 1. Develop a flexible planning model which takes into account catering for different levels.
> 2. Plan a series of activities and the strategies to implement them using the planning format.
> 3. Implement the unit in the classroom and observe.
> 4. Meet regularly as an Action Research Team to discuss the progress of the trial. Modify plans, activities and teaching strategies as appropriate.
> 5. Plan and implement a further unit of work based on the modifications.
> 6. Develop a format for programming a two year cycle of maths using the planning model and based on the levels and strands of the CSF.

Although the model of professional development used in *NPDP Maths* and *Maths in Schools* focused on self-directed professional development in which *all* participants are involved in the decision making processes, it was clear that many teachers did not necessarily want to always be involved. That is, some teachers were content with having major decisions made by the key teachers or others, while they themselves were then the people who implemented these decisions and related ideas. It must, however, be noted that this form of 'top-down' decision making is much different to when a state Department of Education makes decisions that must then be implemented by teachers. In the first case, the decisions are being made by individuals intimately familiar with the school, the teaching situations, and the teachers and students who will be involved, and all the teachers are in a position in which they can question the decisions or processes being recommended. In the latter case, the decision makers are not able to consider the particular circumstances of the school and teachers who will be involved, and teachers are not able to follow-up with any comments or suggestions related to the proposed action plan.

3. Increase in Teachers' Professional Knowledge

Feedback indicated that participants' improved professional understandings were largely a result of the team approach encouraged by NPDP, where teachers had to talk and work together to advance their mathematics curriculum. Nearly every project commented favourably on this collaboration and placed it high on a list of things they would maintain for future professional development endeavours. With regard to positive effects of involvement in NPDP, teachers indicated that they now had increased knowledge of the SOS or CSF and related teaching or assessment skills, and they had developed the knowledge, skills and resources to implement a new mathematics program or new teaching or assessment methods (in line with the SOS or CSF). For example, in the integration of technology into mathematics teaching, or the use of more concrete-based learning experiences.

4. Sharing with Teachers from Other Schools

In both *NPDP Maths* and *Maths in Schools*, full or half day orientation seminars and follow-up feedback and evaluation sessions were held for facilitators and key teachers (and in some cases schools chose to send additional teachers for whom they funded teacher relief time). The objectives of the first seminar were to clarify the goals of NPDP, outline features of the SOS or CSF that might be new to teachers, meet other participants, and exchange ideas about professional development in mathematics education. Oral and written feedback from the sessions indicated these objectives were met. The sessions later in the year were intended to enable participants to share and discuss their project developments and outcomes. Feedback from these latter sessions indicated the participants found the sharing and meeting with other NPDP participants to be a valuable professional development activity.

CONCLUSIONS

Teachers highly valued a model of professional development that was flexible, based on local needs and contexts, and 'owned' by the teachers involved. In particular, they valued:

- the opportunity to work on locally determined teaching problems
- the productivity that results from teachers and university staff working collaboratively
- the collaborative nature of the work conducted by the project team (often whole staff or departments)
- the ongoing nature of the program, so that professional development activities are inter-related and conducted over an extended period of time
- the opportunity to work extensively with a new curriculum document, the *Student Outcome Statements (SOS)*, or the *Curriculum and Standards Framework (CSF)*
- the focus of the professional development on the improvement of teaching and learning

- the flexibility of the professional development model (so that individual projects have tailor made professional development, and the model can be adopted for use in other key learning areas)

Most teachers who were involved in a 3-year evaluation, after their schools were no longer directly involved in NPDP, reported that the subsequent impact of their involvement in NPDP was positive and ongoing. Problematic aspects of the programs over the 3 years included difficulties associated with: scheduling teacher collaborative meeting sessions, maintaining communication between schools or between staff at larger schools, and involving all staff in a way that is non-threatening and supports their individual beliefs, past experiences and current teaching practices.

The guiding curriculum documents in initiation of the NPDP programs are very much in the flavour and spirit of curriculum movements elsewhere in the world (e.g., National Council of Teachers of Mathematics, 1989). They provide a framework around which systems and schools can build their mathematics curricula, acknowledging the changing role of mathematics studies within students' lives and in society in general, and the need for all students to gain access and success in mathematics studies, regardless of gender, social class, cultural background, ethnicity, or geographical location. The goals for school mathematics outlined in the *National Statement* are compatible with the emphases of the *Standards*, including: student development of confidence and competence in applying mathematics in daily situations, development of positive attitudes towards mathematics, capacity to use mathematics in solving problems individually and collaboratively, communicating mathematically, learning use of technology and other tools and techniques that reflect modern mathematics, and experiencing mathematical processes and ways of thinking. Within the culturally diverse population that now comprises most western societies, the nature and role of social and cultural contexts are also emphasised. To achieve any of these goals, an essential first step is teacher professional development. Hence, the NPDP programs have provided a vital component of Australia's efforts to revise mathematics curricula and improve mathematics teaching and learning.

An essential underlying rationale of Australia's National Mathematics Profile is that it respects the professional judgement of teachers. Since the Profile is structured so as to "emphasise the big ideas rather than the details of content and sequence" (Willis, 1994, p. 3), it respects the professional integrity of teachers in relation to facilitating student learning and determining when it has occurred. The Profile is a 'guide' as opposed to a 'recipe' for teaching and assessment because it is descriptive rather than prescriptive. It does not provide explicit advice on how or when to teach particular topics, opting instead to describe "characteristics, behaviours or understandings in the learner which have significance beyond the particular learning sequence or phase, indeed beyond school" (Willis and Kissane, 1995, p. 7). These decisions, as well as decisions related to students' backgrounds, learning styles, readiness for learning, and other personal and contextual factors, are left to the individuals best positioned to make them—classroom teachers. They also do not dictate the explicit behaviours (i.e., as with behavioural objectives) a student must display as indicators that a level has been achieved, allowing a teacher to determine when appropriate depth, breadth and consistency of performance and related factors have been demonstrated.

A document such as the National Profile, in its overall focus on having teachers as the key decision makers with regard to topic, sequencing, teaching approaches and assessment methods, implicitly necessitates teacher professional development. This viewpoint on curriculum development is new, and perhaps threatening, to many teachers. Hence, teacher professional development activities that allow teachers to explore the use of a new curriculum document within a supportive and risk-taking environment are essential to the eventual success of any curriculum changes promoted by a new document. The NPDP *Maths* and *Maths in Schools* have provided such an avenue for professional development and they have been successful in achieving their goals.

REFERENCES

Australian Education Council (1991). *A national statement on mathematics for Australian schools.* Melbourne: Curriculum Corporation.

Australian Education Council (1994). *Mathematics—a curriculum profile for Australian schools.* Melbourne: Curriculum Corporation.

Board of Studies (1995). *Curriculum and standards framework: mathematics.* Melbourne: Board of Studies.

Education Department of Western Australia (1994). *Mathematics student outcome statements, working edition.* Perth: Education Department of Western Australia.

Glaser, B. and Strauss, A. (1967). *The discovery of grounded theory.* Chicago: Aldine Publishing Company.

Kemmis, S. and McTaggert, R. (1988). *The action research planner, 3rd edition.* Geelong: Deakin University Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Powney, J. and Watts, M. (1987). *Interviewing in educational research.* London: Routledge and Kegan Paul.

Willis, S. (1994). Student outcome statements: what's it all about? *Cross section,* 6, 1, 2-7.

Willis, S. and Kissane, B. (1995). *Outcome-based education: a review of the literature.* Perth: Education Department of Western Australia.

Ad Hoc Session 2

MATERIALS TO STIMULATE MATHEMATICAL THINKING AT THE ELEMENTARY LEVEL -
A PROGRESS REPORT IN THE KINDERMATH PROJECT

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In this session, materials were displayed for the after-school enrichment program called Kindermath which takes place in Thunder Bay. Since many of the materials were inspired by the work of other members of the Canadian Mathematics Study Group (CMESG) (Simmt (1996), Davis (1996), Csahoczi (1979) courtesy G. Kondor, and Zack (1995) for example), it seemed only fitting to have a hands-on session at CMESG '97 at which members could view the materials first-hand. And indeed, participants seemed to enjoy trying the activities as much as the children!

Kindermath was described in the 1996 CMESG Proceedings (Kajander, 1996). Briefly it attempts to give children a sense of the beauty and creative potential in mathematics. It is an after-school program operated out of an elementary school, offering eight weeks of one-hour sessions. It attempts to make up for some of the "process deficits" in the curriculum referred to by Schoenfeld (1994, p.58).

Peter Taylor (1997a) says of the mathematics curriculum that it:

> begins with a collection of general (and wonderful) aims, but then ridicules and annihilates these with pages and pages of closely specified technical skills - what we have come to call the LIST (p.5).

Many researchers feel children learn better in a playful and exploratory environment (Davis, 1996; Richards, 1991; Zack, 1995 for example). However, creating such environments need more work, especially at the elementary level where children are both naturally motivated and naturally drawn to create. Such is the goal of Kindermath, which was designed as a possible response to the need for a richer, freer environment in which the children could play in an open-ended way *without* being bound by "the LIST" or by a formal evaluation for grading purposes.

There is a continued emphasis in the *child's own* ideas and creations in Kindermath. The child is free to "make math". The goal of the activity is to give children the experience of actually *doing* mathematical thinking and visualizing.

The following is an outline of some of the activities used in the Kindermath program. These plus about a dozen more similar activities were available at the Conference session for participants to try out—which they did with great enthusiasm!

SAMPLE KINDERMATH ACTIVITIES

Activity	Materials	Description
1. Shape and tower construction with wood pieces.	Wooden pieces about 2cm x 0.5cm x 6cm (cut from purchased lengths	Children play with the pieces, and are encouraged to build towers, various polygons etc. While students work, teacher may form a triangle, square, pentagon, hexagon, etc. Until a student says "that looks like a circle". A discussion of a circle, or any other 'discovered' shape may follow.
2. Square, rectangle, triangle, & parallelogram area measuring	Cardboard pieces: square about 20cm x 20cm, the same square cut diagonally to make two triangles, a parallelogram the , size of the square and the 2 triangles and finally 2 obtuse triangles (a same-sized parallelogram cut diagonally). Also unit squares 5cm x 5cm (about 30)	Again the children are encouraged to investigate the materials, and then to measure the square and triangles with the unit squares. The teacher must encourage them to first measure the square, and then to combine pieces and use this information the measure the triangles (half the square), and then the parallelogram (square plus two triangles) and finally the obtuse triangles (half the parallelogram).
3. Block Building Game	6 cubes: 2 red, 2 yellow, 1 blue, 1 green and 8 clue cards.	Four to eight students work together, each with one or two necessary but incomplete clues. All 8 clues are needed to make the shape. (See McDougall and Kajander, 1997)
4. 3-D Tic Tac Toe	game chips and game boards made of four plexiglass squares about 10cm x 10cm and each with a 4x4 grid drawn on it, mounted vertically.	Students are encouraged to first simply play the game (trying to get 4 in a row in any direction). And then to develop strategies for winning (there are several).
5. Plasticine Shapes	Plasticine and toothpicks	Students make 3-D shapes of their choice; teacher may provide a shape such as a tetrahedron as an example, and discuss its characteristics.
6. Magic Bugs	Pre-cut moebius strips, about 4cm x 80cm, scissors, markers, glue and a plastic bug.	Students are given the 'rules' for the bugs: they can only walk in the surface of the strip, but not cross an edge. They are encouraged to solve the problem by first tracing the path of the bug on one side, and then lifting the strip up off the table and playing with the possibilities.
7. Geometric Design Colouring	Geometric design, fine markers.	Children are encouraged to design a pleasing colour pattern and colour it in. (eg. Burrows, 1992)

8. Tessellations	Paper, scissors, tape, cardboard, square about 4cm x 4cm, markers.	The children are encouraged to design a tessellation pattern of their own from a square, trace it, and colour in the shapes to make a tiling pattern. Discussions about the areas and other properties may follow.
9. Fractals cards	Paper, scissors	Two types of fractal cards are made and examined. While limits are not explicitly discussed, we do discuss whether the contained volumes would fit in a box and how small the box might be.

Kindermath activities also include computer microworlds of various levels of difficulty designed in Logowriter. The simplest of these is a two command microworld , in which F means "go forward a little", and T means "turn" (90 degrees), which were inspired by Rina Cohen at the Ontario Institute for Studies in Education (OISE). The children are encouraged to draw whatever Logo designs they like but are also shown pictures of squares, rectangles, stairs, checkerboards, rectangular spirals, etc. To give them ideas. As the children gain confidence, more commands in full Logo can be given.

Kindermath has been in operation for several years and some initial comments on student reactions to the sessions can now be made.

The children are very enthusiastic. While this may be partly due to the fact that the children choose to come voluntarily, some do also come because their parents choose it for them. Similarly, the children in the regular school classrooms in which these activities have been used are very enthusiastic, and students often ask to borrow the materials for use later in class.

Children seem naturally to think in three dimensions. For example, younger children have been observed to stack blocks 'up' faster than older children who have been used to thinking on a planar surface.

Children rapidly gain ownership of their own ideas. Stan, age 7, completed a geometric shape puzzle in which he had a large triangular space to fill. He had four small equilateral triangle pieces left, and was thrilled when he managed to fit them together to fill the space, calling it "my pattern". Later he was excited to find "his pattern" in a toothpick puzzle, and used it to solve the latter fairly challenging puzzle immediately.

With each session I find myself encouraging the children to do more without me, and they continually surprise me with their capabilities. For example, I used to direct the 'magic bugs' (moebius strip) activity; now I encourage the children to lift the strip off the table and figure it out for themselves—and many do.

It is difficult to say which Kindermath activities children enjoy the most. From informal observation as the teacher/researcher, the cooperative learning tasks such as the Block Building Game (Activity 3) and the other games such as the 3-D Tic Tac Toe (Activity 4) seem to generate even more enthusiasm than the other activities. While some children become extremely excited by the computer, this is not universally the case.

Many of the Kindermath materials have been homemade with a limited budget. There is virtually nothing that could not be constructed by an enthusiastic teacher or group of teachers over a summer holiday. For example, the 3-D Tic Tac Toe games were each made by cutting a square piece of plexiglass in quarters at the lumber store. These were then scored into 4 x 4 grids each with an indelible marker. Finally, purchased doweling lengths (we used about 4 cm pieces) were glued in the corners of each stack of four plexiglass pieces.

Requirements for Kindermath tasks are that they be exploratory in nature and open-ended where possible, and allow for students to try out their own ideas. Activities which illustrate the field of mathematics as an exciting domain in which the child can create imaginative and beautiful ideas of his or her own are the most desirable.

In terms of a broader implementation, many of the Kindermath materials have been successfully field tested in grade one to five classrooms. The children are more responsive if the groups are smaller, and pull out programs work well here if staff is available.

Classroom management and control issues may be difficult for teachers when trying such new materials. McDougall reports that classroom teachers may initially experience a loss of control when beginning to use new exploratory learning environments (1996). He recommends journal writing, mentorships and peer support as possible ways to deal with this initial problem. Experience with the new materials in a classroom setting is possibly the best way to gain confidence.

Teachers need to allow their perception of mathematics to be broadened to include more than "the LIST" of technical skills. Many pre-service teachers I have worked with on activities similar to those used in Kindermath ask "is this really math?" and even more often "how will the children learn math?" (read 'math' = technical skills). Taylor (1997a) asserts that the best way to get the technical skills is to learn them *as necessary* for each particular task. A difficulty with such an approach in the elementary program may be that young children may not have the attention span to remember or remain interested in the original task after learning the skills. But the thoughtful teacher can alter fun and hands-on practice in the technical skills with the creative activities. The technical become more valuable when children can use them to create their own 'works of art'.

The goal in Kindermath is the mathematical experience—*that* is the desired product. Students need to see for themselves how much fun it can be to create a new mathematical idea, without having to strive for "the" (= "our") solution.

In summary, it does appear possible to provide rich experiences for children without a lot of high powered resources. Rather, it is a question of broadening our view of what valid mathematical experiences might be.

We are reminded of Taylor's by now well-known comment that "It's not money that's lacking, not as long as there are enough small napkins around [to write on]. It's courage and imagination." (1996b)

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REFERENCES

Burrows, Roger. *Images*. Philadelphia: Running Press, 1992.

Csahoczi, E., Halmos, I., Radnai, J.S. and Varga, K. *Mathematical Worksheets or Grade 6.* Hungarian Institute of Pedagogy, 1979.

Davis, B. *Teaching Mathematics, Toward a Sound Alternative.* New York: Garland Publishing Inc., 1996.

Gardner, Martin. *The Second Scientific American Book of Mathematical Puzzles and Diversions.* Chicago: University of Chicago Press, 1987.

Jenkins, Gerald, and Wild Anne. *Make Shapes, Series No. 1.* Norfolk England: Tarquin Publications, 1990.

Kajander, Ann. Kindermath: Creating Opportunities for Children to Experience Mathematical Thinking. In review for publication in *Teaching Children Mathematics.*

Kajander, Ann. Mathematics Enrichment in an Out-of School Setting. In Y. Pothier (Ed.) *Proceedings of the 1996 Annual Meeting of the Canadian Mathematics Education Study Group.* Halifax: Mount Saint Vincent University Press, 1996.

McDougall, D. and Kajander A. Enriching Mathematical Creativity in the Elementary Classroom. *Ontario Association of Mathematics Educators Gazette,* vol. 36 #2 Dec. 1997.

McDougall, D. *Mathematics Teacher's Needs in Dynamic Geometric Environments: A Search for Control.* Unpublished Doctoral Thesis, Toronto: University of Toronto, 1997. Ministry of Education and Training. *The Ontario Curriculum, Grades 1-8, Mathematics.* 1997

Richards, J. Mathematics Discussions. In E. Von Glaserfeld (Ed.), *Radical Construction in Mathematics Education.* Boston , MA: Kluwer Academic Publishers, 1991, pp 13-51.

Schoenfeld, Alan H. What do we know about Mathematics curricula? *Journal of Mathematical Behaviour.* 13, 55-80, 1994.

Simmt, E. *Parents & Children & Mathematics: Occasioning in Mathematics Cognition.* New York: American Education Research Association, 1996

Taylor, Peter. *A Senior High School Mathematics Textbook - In Progress.* Kingston: Queen's University Press, May 1997a.

Taylor, Peter. Needed: Smart and Small Napkins. *The Globe and Mail.* Toronto: February 6, 1997b.

Van Cleave, Janice. *Math for Every Kid.* Toronto: John Wiley and Sons, Inc. 1991.

Zack, Y. Algebraic thinking in the upper elementary school: The role of collaboration in making meaning of 'generalization'. *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education.* Recife, Brazil, 1995, pp 106-113.

Ad Hoc Session 3

TOMORROW'S MATHEMATICS CLASSROOM: A VISION OF MATHEMATICS EDUCATION

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The National Mathematics Education Institute, based at Queen's University is, as its title suggests, a national organization dedicated to the reform of mathematics education in Canada. Most participants at the first two annual meetings of the Institute, held in the summers of 1994 and 1995 (and patterned on annual meetings of the CMESG) came to the conclusion that one major obstacle to reform in mathematics education lies in a weak or incoherent vision of what a mathematics classroom should be like. They also concluded that this vision would have to include a clearer understanding of the nature of the learner, the nature of the teacher, the nature of mathematics as embedded in rich learning tasks, and the nature of the interaction among the three. A smaller group of people met at the third annual meeting in 1996 to develop such a vision (in order to "clarify issues and to stimulate debate on desirable directions for curriculum reform in Canada.")

At the CMESG ad hoc session, the members listed above described four examples of this vision. They were constructed around four learning activities, one for each of the primary grades, the junior grades , the middle school grades and the senior grades. Each manifestation of this vision was (or at the time of the ad hoc session, about to be) printed as a 6-panel, 4-colour, 3-page foldout brochure. The brochure said to the reader this is what the vision looks like in the classroom, this is what the math looks and feels like, this is what students do when they learn math and this is what teachers do when they help students learn. Contents of the brochures, their common features, the vision contained within the brochures, and possible uses of the brochures was discussed by those who participated in the ad hoc session. Two of the brochures were distributed (Junior:"Tracing Tracks"; Senior:"The Two Trees Problem").

We conclude this brief description of the ad hoc session with a restatement of the common last page of each vision brochure, a statement of the vision of tomorrow's mathematics classroom under the titles Mathematics, The Student, and The Teacher. For further information on these brochures or other activities of the National Mathematics Education Institute, contact Bill Higginson at Queen's. Study Group members wishing to obtain one set of the brochures should indicate this in an e-mail sent to 'higginsw@educ.queensu. ca'. Larger quantities are available at cost. This ranges from $1 per brochure for small orders to $0.80 and $0.60 for larger and 'very large' orders.

VISION STATEMENT

In Tomorrow's Mathematics Classroom:

Mathematics:

- is experienced as a diverse, powerful, and evolving discipline, as a way of thinking, as a way of communicating, and as a way of perceiving the world, with significant links to all aspects of human experience;
- emerges from, and is made explicit through, exploration and interaction, using a wide range of technologies and resource materials;
- is embedded in potentially-rich learning situations that are interesting and relevant for students and enable all to participate and grow;
- plays three important roles; as a set of useful tools, as one of many disciplines which can contribute to the understanding of a situation, and as a field worthy of study for its own sake, that is, as servant, citizen and sovereign.

The Teacher:

- provides students with stimulating and well-designed learning activities to promote intellectual, emotional and social growth;
- interacts with students to encourage, inspire, challenge, discuss, share, clarify, articulate, reflect, assess, and to celebrate growth and diversity;
- shows the benefits that come from keeping abreast of developments in mathematics and mathematics education;
- acts as someone who assists, someone who confirms and directs, and someone who animates and inspires students by epitomizing the curious, enthusiastic, passionate, and risk-taking learner, that is, as informer, facilitator and artist.

The Students:

- build their own mathematical knowledge through a process of exploration, interaction, and reflection, centered on rich learning activities;
- develop and refine skills in the areas of mathematics, communications, problem-solving, logical reasoning, creative thinking, technology, independence and interdependence;
- use their skills to deal effectively, confidently, sensitively and objectively with situations involving complexity, constraints, diversity, novelty, ambiguity, uncertainty, and error;
- act as individuals who select and use existing rules, understand the principles and patterns underlying these rules, and who create new rules to deal more effectively with situations, that is, as compliers, cognizers, and creators.

Ad Hoc Session 4

A MODEL FOR THE DEVELOPMENT OF ALGEBRAIC THINKING

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INTRODUCTION

Mathematics, as a human activity, is a particular way of thinking and communicating: A mathematician is thoughtful. Mathematically, mathematics is a language of relationships, pâtterns, and inferences. It is not a body of ideas, but is a study of thinking.

When we study mathematics, we see that algebra is the essence of mathematics. In fact, algebra is a language of mathematics and power (mathematical power). It is a way of understanding our real world. Dunne and Jennings (1996) consider that "the essence of mathematics is algebra," and they suggest that "algebra is central to using and applying mathematics." [3]

The National Council of Teachers of Mathematics (1989) [4], presents a particular vision of school mathematics. It is a framework to developing and responding to changes in mathematics curriculum. In the NCTM standards, a great deal of emphasis is placed on mathematical thinking as algebraic, geometric, and probability reasoning. In Grades 5 to 8, the mathematics curriculum should include explorations of algebraic concepts and processes so that students can understand the concepts of variables, expressions, and equations.[4]

This paper presents a model for the development of algebraic thinking. The model supposes that thinking in algebra develops through a sequence of five levels. The levels begin with intuitive or familiar, and end with symbolic or formal thinking.

RATIONALE

The difference between the pedagogy of algebra in different times, grades, and countries is a difference in our vision of the nature of mathematics and how mathematics is taught. Early in the twentieth century, algebra was various systems of linear equations, algebraic methods and algorithms. Learning algebra focused on the mastery of algebraic skills. The main objective was to gain skill in algebra. In the 1960's, through "new math" movement, the focus of mathematics curricula began to shift toward investigating and understanding algebraic structures. Since the 1990's, algebra structure and language has been changing and new topics, such as computer based methods and mathematical modeling have been added to mathematics curriculum.

The difference between this view is the difference between our vision of the nature of mathematics as a formal system and mathematics as a particular way of thinking. The difference between the record of thinking (as a structured body), and thinking-way is the difference between the formal vision and the constructivist vision of mathematical knowledge as Romberg. [6]

In the formal vision, the mathematical knowledge is a structured body [1] ; structured and fixed, which is the science of mathematics. Thus, mathematics is a logical body of facts and ideas, that is, a discipline. In this view point, algebra is a fixed body of experience that contains equations, polynomials, functions, axioms and theorems. The number systems are central to studying properties and proving rules and theorems that have been abstracted from the body. Abstracted experience is a good guide to making something, that is, mathematics. In the constructivist vision, mathematics knowledge is an activity of re-constructing experience in a successful and powerful way as discovering patterns, constructing relationships, making inferences, problem solving, proving theorems and modeling. I see mathematics in the way of thinking. Then, algebra is a language and power of mathematics. As Wagner and Parker (1993) state, "algebra is a language for describing actions on and relationships among quantities."[7]

The primary goal of algebra is to:

- develop algebraic reasoning
- understand language of patterns
- study structures of number systems
- problem-solving.

In the higher grades, algebra is central to constructing mathematical modeling and studying axiomatic structures. In short, when we are studying mathematics, we see that algebra is a language of mathematics and mathematical power. The explicit message is that algebra is a way of studying patterns and mathematical modeling. When students have developed some confidence in using symbolic reasoning, they see mathematics as a real language.

Algebraic thinking is a type of mathematical thinking that involves constructing relationships and patterns of quantitative variables, conjectures and symbolic reasoning, proving algebraic theorems through using inference and understanding and developing a logical system based on axioms.

Because algebra is a symbolic language, symbolic reasoning is the essence of algebraic thinking. I think that algebraic thinking, as geometric thinking or proportional reasoning and probability thinking, starts intuitively in students' activities and in their own way and develops over time into formal symbolism, then at the deductive level. I see that intuitive ideas about patterns, variables and conjectures are the critical points (factors) in achievement of algebraic thinking.

A MODEL OF ALGEBRAIC THINKING (MAT)

A Model of Algebraic Thinking (MAT) is a vision for understanding how students think algebraically. The goals of the model are (1) to provide a framework for developing reasoning in algebra; (2) phases and processes to facilitate the understanding of the nature of thinking and the students' ways in algebra; and (3) to provide a base for developing teaching-learning tasks in the algebra classroom. In general, MAT is a language that is explorative and hierarchical for describing phases of reasoning in algebra.

MAT has been formulated according to current thought on mathematics education, such as the van Hieles' perspective of reasoning in geometry [2]. The principal assumption in the model is that reasoning in algebra develops through sequential levels from the intuitive and familiar to the formal. MAT consists of five levels of reasoning in algebra: (1) intuitive level, (2) induction (familiar) level, (3) abstraction level, (4) deduction level, and (5) formal level. The characteristics of each level are given below:

Level	Characteristics, process
Intuitive	• exploratory, manipulative experiences • free to conjecture

	* relationships among objects
Induction	* familiar representations * symbolic manipulations * generalizations
Abstractions	* abstract concepts and expressions * higher abstractions of different functions
Deduction	* understanding of formal proofs
Formal	* study of axiomatic systems in algebra * different models

THE LEVELS OF ALGEBRAIC REASONING

The first level of algebraic reasoning is intuitive thought. In free-discovery and explorative settings, the students begin to learn pre-algebra concepts by their intuitions. The students reason about the relationships between and among real objects by their awareness and their expectations. In general, the students reason about various objects such as time, rate, distance and prices in intuitive ways more than in symbolic ways. Many elementary school students have difficulty using symbolic reasoning.

If intuitive thought, or intuitive notions about relationships, is the natural approach to learning algebra, then the symbolic reasoning or symbolic-induction reasoning is the essence of algebraic reasoning. Thinking about numbers and the use of symbols to represent different objects are basic characteristics of the second level. The induction level is pre-algebra level as a link between numbers and symbols. In this level, the students reason about the use of symbols as abstract objects, unknown quantities and variables or as numbers The students use symbols in formulating patterns, relationships, or for symbolizing mathematical generalizations. In short, this level contains pre-algebra reasoning and appears to be the critical entry level as the van Hieles' level 2 of geometric analysis. [5]

The third level reflects abstract thinking in algebra. It is a transition from familiar experiences to formal methods in algebra. In short, it is a transition from familiar to abstract. At this level, students introduce the abstract forms of concepts of algebra such as definitions and characteristics of functions, algebraic entity, solution of equations, operations on polynomials and equivalent expressions. Thus, students are able to learn algebraic methods and apply problem solving in algebra. Students sometimes have difficulty with more deductive proofs but they can prove a direct argument by direct proof.

The fourth level reflects deductive thinking in algebra. It is a study of formal proofs or logic, in algebra. At this level, students are able to understand the deductive nature of mathematics; algebra or geometry. Thus, more students are able to construct conjectures, prove theorems, understand and use axioms, and understand direct and/or indirect proofs. In general, students are able to do deductive proofs by using the various components of axiomatic algebraic structures such as undefined terms, axioms, theorems and definitions. Because the axiomatic structures are a formal logical system, students can learn deductive mathematics in high school and college.

The fifth level reflects understanding of formal algebra; the structure of algebra is more formal and models of algebra are various systems. At this level, students reason about logical systems and are able to understand algebraically the difference between axiomatic systems. This level is problematic for most secondary school students and is, therefore, more suitable for university study.

According to each level, the appropriate mathematics experiences, concrete or abstract, intuitive notions or concepts, numbers or variables, are important tools to help teachers and students reflect on the levels.

SUMMARY

Mathematics is a way of thinking and communicating; algebra is a language of mathematics and mathematical power. Algebraic thinking is a type of learning algebra. Algebraic thinking, as process, develops through a sequence of five levels: intuitive, induction, abstraction, deduction, and formal. Most students need appropriate experiences for developing algebraic reasoning at all levels.

REFERENCES

1) Ariel, A. (1992). *Education of Children and Adolescents with Learning Disabilities*. New York: MacMillan.

2) Burger, W. F. and Barbara, C. (1993). Restructuring geometry. In Patricia S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics*. p. 140-154. New York: MacMillan.

3) Dunne, R. and Jennings, S. (1996). A revised view of the nature of mathematics. *Mathematics Teaching*, 156, 30-31.

4) National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: The Council.

5) O'Daffer, P. G. and Thomquist, B. A. (1993). Critical thinking, mathematical reasoning and proof. In Patricia S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics*. p. 30-50. New York: Macmillan

6) Romberg, T. A. (1992). Problematic features of the school mathematics curriculum. In Philip W. Jackson (Ed.) *Handbook of Research on Curriculum*. p. 749-788. New York: Macmillan.

7) Wagner, S. and Parker, S. (1993). Advancing algebra. In Patricia S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics*. p. 119-139. New York: Macmillan.

Ad Hoc Session 5

WORKING TOWARDS CURRICULUM RENEWAL IN UNDERGRADUATE MATHEMATICS

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INTRODUCTION

This report summarises two related projects that explore undergraduate mathematics teaching and learning. A number of factors led to development of these projects, including: (i) the need for a major evaluation of the mathematics units intended to service students in other departments (e.g., engineering and science departments), (ii) the changed background skills of first-year students as a result of state-wide changes in secondary mathematics curricula, (iii) the more diverse student cohort in relation to educational, cultural, and socio-economic backgrounds, (iv) the upcoming mandated use of graphics calculators in secondary mathematics courses, and (v) changes in the employment environment of graduates, particularly in relation to the increasing prevalence of sophisticated technological aids.

The main aim of the first study was:

A. To investigate the role and relevance of undergraduate mathematics studies as perceived by students and lecturers in science and engineering.

Results of this study provided avenues of consideration for development of the second study, which is a five-year curriculum renewal project in undergraduate mathematics. Overall, the curriculum renewal project aims to develop the teaching of undergraduate mathematics courses through identifying deficiencies in the mathematical background of entering students, utilising technology in lectures (such as computer-generated presentations and graphics calculators), and developing teaching and assessment practices that more fully meet the needs of the diverse range of students now studying mathematics. The project, starting with first-year students, would progress to other years over time. Initial stages of this five-year process are reported here, and specifically, results related to the following four research objectives:

B1. To determine the strengths and weaknesses of first-year science and engineering students' mathematics backgrounds.

B2. To identify the level of confidence students have in their mathematics knowledge and skills.

B3. To determine reasons for students' perceived lack of confidence in their mathematics background or current course work.

B4. To determine faculty attitudes towards current and proposed curricula, specifically in relation to assessment changes and increased use of technology.

FRAMEWORK

Four areas of research were useful in formulating the research. First, the world-wide calls for reform in mathematics education from a variety of sources (Apple, 1995; Dubinsky, et al., 1994; National Research Council, 1991; National Research Council, 1994). Second, the role of affect in mathematics learning, the effects of which are now clearly acknowledged (for example, see Leder, 1993). Third, the ideas of change processes as they apply to education. We chose to focus on faculty attitudes towards change because others (Elmore, 1995; Hargreaves, 1994) had shown curriculum renewal efforts to be dependent upon the willingness of teaching faculty to accept change. Fourth, notions related to integrated curriculum, which links the three components of syllabus, pedagogy, and assessment. Although the importance of this link has been highlighted by many educators (Burton, 1992; Parker, 1995) it has often been ignored in curriculum development projects (Burton, 1992). Consequently, professional development for faculty is essential in order to explicate this link and help faculty enact a reformed curriculum. The National Research Council (1994) also emphasised the need for faculty in higher education to participate in ongoing professional development concerned with both their disciplinary and teaching expertise.

METHOD AND DATA SOURCES

Part A. Perceptions of the Role and Relevance of Undergraduate Mathematics

Part A of this research was concerned with both identification and description of individuals' perceptions of experiences, and therefore employed a blend of quantitative and qualitative research methods. In particular, the study was a survey project, with follow-up interviews to elaborate and broaden the survey findings. Survey data was analysed and summarised in terms of prominent categories and patterns, while an inductive reasoning approach for data analysis was adopted for the interview data (Glaser & Strauss, 1967; Powney & Watts, 1987). Specifically, the data sources and data collection methods used the followed outline.

> (a) Survey: A survey questionnaire on views of relevant mathematics and the role and relevance of mathematics in university programs and later related employment was administered to 306 students enrolled IN a first semester undergraduate mathematics course, 151 students enrolled in their final year of undergraduate studies, and 38 lecturers from a range of disciplines whose undergraduate programs require some mathematics studies.
>
> (b) Interviews: A group of 20 individuals comprised this more elaborated component of the study; 6 first-year mathematics students, 6 final year students, and 8 lecturers, all of whom were in the fields of physical or natural science, computing science, or engineering (the programs which require a first-year course in calculus and linear algebra, and perhaps some statistics). The interviews were semi-structured in nature and the interview protocol was based on the questionnaire items, but with an aim to obtain more detailed and elaborated responses.

Part B. Curriculum Renewal Project

Part B of the research (first phases of the curriculum renewal project) was primarily concerned with identification of factors related to students' skills and experiences and faculty attitudes, and therefore used surveys, a diagnostic assessment instrument, and discussions with workshop and project leaders. All 23 teaching faculty and 8 part-time tutors in the mathematics department, along with 350 undergraduate science or engineering students enrolled in a first or second semester course in calculus and linear algebra were surveyed.

To address the first research objective, a twenty-item multiple-choice diagnostic test was developed. The items were designed to assess students' knowledge and skills in those topics considered essential background for these first-year courses. Students completed the test before the beginning of the semester. In addition to providing data for this research, the results of the diagnostic test were used in the context of the larger curriculum development project to inform students of their weaknesses and direct them to appropriate remedial materials.

To address the next two research objectives, a forty-six item questionnaire was developed which asked students to respond about their level of confidence with their mathematics background and their current mathematics course work. Responses were recorded on a four-point Likert scale ranging from "Very Confident" to "Very Unconfident", with an additional response point for students to indicate if they had not studied the topic. Two open-ended items were included to elicit responses about the reasons for any lack of confidence. The survey was conducted towards the end of second semester 1995.

To address the fourth research objective, a fifteen-item questionnaire was designed to assess actual and preferred use by faculty of a range of reformed teaching and assessment methods, including their use of various forms of technology. Each question required two responses: one response about the actual frequency of occurrence in the respondent's teaching, and the other response about the preferred frequency. Responses were recorded on a dual five-point Likert scale ranging from "Very Often" to "Almost Never". An afternoon workshop, the first in a series, was held to work with faculty on issues related to students' backgrounds and learning needs, as well as current trends in mathematics assessment practices. Subsequent workshops were held to familiarise faculty with the technology available for use in their classes, and to discuss issues pertaining to the use of tutorial time. After each workshop, the presenters wrote a few paragraphs of reflections about the workshop documenting their perceptions about its success.

RESULTS

Part A. Perceptions of the Role and Relevance of Undergraduate Mathematics

The results from both the questionnaires and interviews indicated that students and lecturers see mathematics as a very important subject of study, although they sometimes questioned the value of particular topics, theorems, or procedures. Across the various groups a number of common themes or issues emerged: relevance, time constraints, teaching issues, role of technology, content choices, and inter-communication between university departments. More detailed descriptions of these issues than are provided here can be found in Frid (1996).

Relevance

More than 90% of the nearly 500 individuals surveyed or interviewed mentioned the issue of the relevance or application of mathematics as a primary factor in its value as a subject to be taught at post-secondary level. That is, they emphasised the need for mathematics to be taught as an applied discipline in which students learn how to use mathematical ideas and techniques to solve real world problems encountered by a range of professionals.

Many of the lecturers stressed that the mathematics actually used within their disciplines is somewhat different in nature to what is taught in many undergraduate courses. They saw this difference as arising from the fact that much use of mathematics requires mathematical modelling processes, for which high proficiency with a wide range of mathematical skills is not enough for success. Rather, success demands one work flexibly, sometimes using educated, experience-derived 'guesses' and a cycle of testing and revision of mathematical models. Both the students and lecturers commented that some people had difficulties in some upper year science or engineering courses because they had achieved success in performing mathematical skills but had not grasped the underlying concepts and how the related ideas are manifested or interpreted in relevant, real world, applied contexts. Thus, the issues of understanding mathematical topics and "having a feel" for them were also often commented on by lecturers.

Time Constraints

The issue of time constraints was mentioned by a majority of subjects (>80%). They expressed concern about what they saw as an 'overcrowded' mathematics curriculum. In particular, they saw the number of topics and related skills covered in undergraduate courses as excessive in relation to the time allocated to teaching them. They felt this situation was not appropriate for supporting students to learn with understanding of concepts and the ability to apply these concepts flexibly. A number of people expressed views that an educationally beneficial alternative would be to focus efforts on understanding ideas and how to apply ideas, while simultaneously taking emphasis away from time consuming techniques that can now be done quickly and easily with the aid of calculators or computers.

Teaching and Learning Issues

Both in the comments made on the questionnaires and in the interviews people diverted from the focus of the questions (mathematics content) to related teaching issues. That is, without direction to do so, they naturally expressed a view that how things are taught is perhaps more important than the details of what is taught. They were of the view that mathematics teaching needed to be more fun, engaging the learner in meaningful, interesting, intellectual activity.

Role of Technology

There were mixed views on what role technology (specifically, graphing/programmable calculators and computer software programs) should take in mathematics teaching. Some first year students were uncertain as to whether or not technology would help or hinder learning, while most other people expressed views that it is essential that learning to appropriately use technology be an integral part of any undergraduate mathematics program. Their views are summed up by the statement: "These things are out there available for use in the workplace, so we might as well teach students how to properly use theM as tools to assist thinking and problem solving."

Inter-Department Communication

The role of communication between science or engineering departments and a mathematics department (or whoever is responsible for undergraduate mathematics teaching) was an issue introduced by individuals within each of the groups. They said that better communication needed to be established by all groups concerned with undergraduate mathematics education so that they could better identify the needs of various departments, as well as the constraints under which a course is offered. This sort of network would facilitate curriculum change that could address the other issues raised: relevance, time constraints, teaching issues, and the role of technology.

Content

People's views on what was appropriate content for first-year mathematics studies were extremely varied, with this variance related to the particular subject discipline in which they worked. For example, a biology lecturer felt her students "must be able to plan a research study and collect data which requires statistical procedures and graphical summary." In comparison, electrical engineers felt mathematics courses should "relate specifically to electrical engineering, not the masses" (fourth year electrical engineer). In general, students were quite vocal about specifying what they thought should or should not be included in required mathematics courses, saying such things as: "Do not do the stuff irrelevant to chemical engineering (e.g., triple integrals and polar coordinates)" and "Gear it towards each course (e.g., chem engineers less dealing with line integrals, matrices, vectors). More dealing with relevant parts such as problem solving" (two fourth year chemical engineers). The general view of students was that mathematics courses should be geared towards the needs of their future professions.

Part B. Curriculum Renewal Project

Student Diagnostic Tests and Remedial Materials

The student diagnostic test has now been run twice (early 1996 and early 1997). The first diagnostic test in 1996 was based on a test developed by Stephen Hibberd at the University of Nottingham (Hibberd, 1995). It was designed for students who have completed the British A-level mathematics, and as a consequence was aimed at much too high a level. Due to the way in which it was set up and implemented, there was no way of changing the content. It was a good pilot test for the second year which has proved to be much more productive. For the second test, the questions were written by faculty in the department, and were at a more appropriate level in terms of the students' skills. This new test (called MQUEST) is available on the Internet and students can log on to do a test whenever they want from any appropriately connected computer terminal. Additional details on the development and use of MQUEST can be found in Caccetta, Hollis, Siew and White (1997).

Since introduction of the diagnostic test, enrolment has increased in courses that provide extra lecture and tutorial time to help students make up deficiencies in their mathematics backgrounds. Students who need it have also been directed towards printed and computer-based remedial materials. These materials were used effectively by some students throughout the first year of the project. During the second year (1997) the use of remedial materials has become more formalised through their incorporation into tutorials.

Remedying deficiencies is considered as an essential first step in curriculum renewal because first-year mathematics courses are structured around large group lecture presentations supplemented with smaller group tutorials. Further, teaching is predominantly guided by textbook formats and practice exercises, and assessment is predominantly examination based. Students with mathematical deficiencies struggle in such an environment, and unfortunately, due to cutbacks in university funding, this lecture/tutorial structure is not likely to change. However, efforts are underway to provide all students with self-learning material designed to enhance lecture presentations. A number of self-learning modules have been written to assist students in first semester mathematics courses. In addition, current efforts are being directed towards the development of technology based tools to create an improved learning environment for all first year students. This will enable students to work through modules at their own pace, along with providing them with self-assessment tests for these modules.

Student Survey

From the survey given to students in late 1995 it was found that, regarding their mathematics background, students had a high level of confidence. Students were least confident with areas related to use of technology, specifically use of graphics calculators and mathematics computer software packages. Many students indicated that they had had little experience of these areas in their school studies. Students were reasonably confident with basic algebraic skills, but were less confident with graphing techniques and interpreting written mathematics explanations and problems. For current mathematics course work, students were least confident with those topics in which they had had the least background preparation in high school mathematics courses, namely sequences and series, and Taylor and Maclaurin series. The open-ended responses, discussed below, illuminated students' perceived reasons for difficulty with current course work.

Open-Ended Responses

There were two open-ended questions at the end of the student survey. The first asked students to list reasons why they lacked confidence with their current course material. The second asked them for suggestions as to how these courses could be improved in future. The responses from both questions have been grouped into three areas: course material, delivery, and student skills and background. It is noteworthy that these three main areas overlapped strongly with areas of concern that emerged with a different sample of students (in Part A of the research).

Course material

Many of the students said they had weaknesses in their backgrounds that led to present difficulties. Sequences and series, and Taylor and Maclaurin series were most frequently mentioned as causing problems. These are done late in the course, and students found the concepts difficult. Many just don't bother to study these topics as they feel the cost to outweigh the benefits. Some commented that there was too much repetition of high school material and not enough time spent on new material. Some commented about the number of formulae to remember. Many commented that they could not see the relevance of much of the course material to their own major course of study. A few mentioned that the text was hard to understand and should be updated.

Delivery

Most of the comments about the delivery of courses were about the speed of lectures, the amount of material students were expected to copy down in one lecture, and the difficulty of simultaneously copying and listening to the lecturer explain concepts. They would have liked to have had the course notes available in some accessible printed form. Many commented that they would have liked more practical and relevant (and in some cases more complicated) examples in lectures, and would have liked more tutorial time and more examples worked by the tutor during tutorials. A few mentioned that in the bigger lecture theatres a microphone is needed, and that the white board in some of the lecture theatres is hard to see. Some requested that the assessment structure be changed to include more course work and less weighting on the exam. Some suggested that using graphics calculators and computer software packages would enhance understanding.

Student skills and background

Those students who didn't have the prerequisites or had not recently studied maths had considerable problems with the speed of the course. A few requested more lecture time. Many cited the number of courses they were required to take in their programs as reasons for not having sufficient enough time to study mathematics as much as they needed. Mathematics is not a priority for them. If they are sick, miss lectures or otherwise fall behind, it is very hard for them to catch up. Lecture notes and worked examples provided before the lectures, or at least in printed form after the lecture, would greatly facilitate the making up of work. A number of these factors overlap with those that have been categorised under course material and delivery, indicating that there is interplay between various aspects of students' learning.

Staff Surveys

The first staff survey was given to all teaching staff in the mathematics department. With such a small proportion of the staff involved with first-year units, we were pleased with the number of responses.

The second survey was given in March 1997 to only those staff involved with teaching first-year courses. Names were optional. Each response was given a score to facilitate analysis, with "Almost Never" given a score of 1, "Seldom" a score of 2, "Sometimes" a score of 3, "Often" a score of 4, and "Very Often" a score of 5. The lowest frequency score was 1 rather than zero, as non-responses were not included in the calculation of means. The mean scores for actual and preferred use, as well as the differences between means, were calculated for each survey (see Table 1). Due to the low numbers completing the survey, further statistical analysis was not feasible.

We viewed the results of this survey quite positively, with the highest differences between preferred and actual use in 1995 in the areas of assessment (individual project, group projects, student assessments, investigations) and use of technology (graphics calculators and computer demonstrations). This indicated that many of the staff saw a need for change. The most worrying feature of the 1995 responses was the low actual use of graphics calculators, given that the use of graphics calculators was soon to become mandatory in secondary school examinations. On the 1997 survey staff reported using more of all the listed reformed teaching and assessment practices except 'individual projects' and 'negotiation of assessment type and weight'. Clearly, some changes had taken place.

Table 1

Mean scores for staff survey ratings of use of teaching and assessment methods

Teaching and assessment methods	Actual '95 n=13	Pref. '95	Diff '95	Actual '97 n=5	Prior to '97	Diff '97-prior
Scientific calculators	3.0	3.1	0.1	4.2	4.2	0
Graphics calculators	1.6	2.5	0.9	1.8	1.4	0.4
Videos	1.6	2.3	0.7	2.0	1.4	0.6
Computer (demos)	2.1	3.1	1.0	3.0	2.4	0.6
Computer (student)	2.6	3.3	0.7	3.2	3.0	0.2
Individual projects	2.7	3.9	1.2	3.4	3.8	-0.4
Group projects	1.4	2.3	0.9	2.6	2.2	0.4
Investigations	2.0	2.8	0.8	3.2	2.8	0.4
Student presentations	1.8	3.0	1.2	3.4	3.0	0.6
Negotiation of assessment type	1.7	2.1	0.4	1.2	1.2	0
Negotiation of assessment weights	1.6	1.9	0.3	1.4	1.4	0
Library tasks	1.9	2.6	0.7	2.2	2.0	0.2
Written tasks	3.4	3.8	0.4	4.0	3.8	0.2

Staff Development Initiatives

The first workshop (late 1995) focused on the background experiences of first-year students. The content and sequence of secondary school mathematics classes were outlined with implications for first-year students being discussed. A small investigation, typical of the type of activity encountered by students in secondary school, was given to the faculty during this workshop. From workshop feedback and from the reflections of workshop leaders, it was clear that faculty were unaware of the mathematics background of the majority of their students, and they were unfamiliar with the types of activities encountered by students in secondary mathematics classes. These activities include open-ended investigations, group projects, writing projects and problem-solving exercises. Although many first-year students currently had some experience with technology in their secondary mathematics courses, faculty made limited use of technology in their current teaching practices, whether it be sophisticated graphics calculators or computer software packages, or more readily accessible scientific calculators.

The second and third workshops (1996) focused on familiarising staff with technology such as graphics calculators, software such as Power Point, and multi-media presentations modes using CD ROMs. The fourth workshop (1997) was an open forum for staff and students to discuss the use of tutorial time. There were presentations from staff and students with a considerable degree of discussion generated. The main concern seemed to be the difficulty tutors have in dealing with the wide variety of students' capabilities. No consensus was reached but this topic certainly has implications for staff development.

In the current phase of the five year project efforts are being made to assist staff in integration of appropriate software packages into their teaching practices. Commercial software packages that are being considered include Maple, MATLAB, Derive and Mathematica. It is expected that this aspect of the project will take considerable time to develop and fully implement. To keep up with the technological advancements will require staff to rethink their notions about how mathematics should be taught, as well as how mathematics curricula can be made more relevant. These features of the impact of technology are a main component of a need to provide support for the teaching and learning of undergraduate mathematics.

CONCLUSIONS

There are a number of issues concerning curricula that need to be addressed by curriculum developers. First, is the issue of how much content is really needed in undergraduate mathematics courses. Many courses are so crammed with content, that instructors naturally use the easiest and fastest mode of transmitting information, namely large traditional-type lectures that assume knowledge is a knowable truth that can be transmitted to passive dependent students. The speed at which students perceive that material is presented to them was the most prevalent issue students commented upon. Clearly, if students see lectures in their present format as ineffective and inefficient, then lectures are not successful in assisting students in their mathematics learning. Alternative and additional avenues for instruction and learning need to be explored.

Second, is how to incorporate relevant practical real-world applications into all areas of mathematics: students have the right to expect that what they are learning has some relevance to their other courses, and staff have the obligation to help them make those connections. The main criticism students had of course material was that it did not connect to their other studies and in particular did not give them any sense of where the concepts and skills could be applied in real world, practical situations. How to use contexts that are applicable to all students' interests and experiences is a challenge curriculum developers will have to face.

Third, is how to better communicate between departments the requirements for students' course work so as to avoid overloading: mathematics departments provide service courses for many other departments at the university and curriculum developers need to be aware of the course loads of those students. Communication about the nature of the mathematics used in other disciplines is also important, as highlighted in Frid (1996).

Fourth are considerations as to how to better prepare students for a technological society: new approaches to mathematics require new syllabi, pedagogical and assessment practices. Students are expected to know how to use computer graphing packages, for example, in courses other than mathematics. It remains to be decided as to whether it is the responsibility of the mathematics department to teach these skills. These new approaches will not happen by themselves, and curriculum developers must give due thought and consideration as to how to implement change.

Fifth, is how to ensure university staff are more informed of the mathematics background of their students: they should both avoid repeating content adequately covered in high school, and make sure that there are no gaps in areas that are essential pre-requisites for understanding. Last, is how to improve the awareness of staff of current trends in mathematics education at all levels: stronger links need to be made between schools and universities.

REFERENCES

Apple, M. W. (1995). Taking power seriously: New directions for equity in mathematics education and beyond. In W. G. Secada, E. Fennema, and L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 329-348). Cambridge: Cambridge University Press.

Burton, L. (1995). Moving towards a feminist epistemology of mathematics. In P. Rogers and G. Kaiser (Eds.), *Equity in mathematics education: Influences of feminism and culture* (pp. 209-225). London: The Falmer Press.

Caccetta, L., Hollis, P., Siew, P.F., & White, B. (1997). Computer based diagnostic testing. In N. Scott and H. Hollingsworth (Eds.), *Mathematics creating the future, Proceedings of the 16th Biennial Conference of the Australian Association of Mathematics Teachers* (pp.66-70). Melbourne: Australian Association of Mathematics Teachers.

Dubinsky, E., Schoenfeld, A., and Kaput, J. (Ed.). (1994). *Research in collegiate mathematics education.* Rhode Island: American Mathematical Society.

Elmore, R. F. (1995). Structural reform and educational practice. *Educational Researcher, 24*(9), 23-26.

Frid, S. (1996).*Undergraduate mathematics: perceptions of its role and relevance in students' lives.* Paper presented at the Annual Meeting of the American Educational Research Association, New York.

Glaser, B. and Strauss, A. (1967). *The discovery of grounded theory.* Chicago: Aldine Publishing Company.

Hargreaves, A. (1994). *Changing teachers, changing times.* London: Cassell.

Hibberd, S. (1995). The mathematical assessment of students entering university courses. In L. Mustoe and S. Hibberd (Eds.), *The Mathematical Education of Engineers* (pp. 313-329). Oxford University Press.

Leder, G. (1993). Reconciling affective and cognitive aspects of mathematics learning: Reality or pious hope? In Hirabayashi, I., Nohda, N., Shigematsu, K. and Lin, F. (Eds.), *Proceedings of the 17th International Conference of Psychology of Mathematics Education.* Japan: Program Committee of the Seventeenth International Conference for the Psychology of Mathematics Education.

National Research Council (1991). *Moving beyond myths: Revitalizing undergraduate mathematics.* Washington, DC: National Academy Press.

National Research Council (1994). *Achieving quality in undergraduate mathematics.* Washington, DC: National Academy Press.

Parker, L. H. (1995). *Curriculum renewal in upper secondary mathematics: the limits and possibilities of reform. Keynote.* Paper presented at the Annual Conference of the New Zealand Association of Mathematics Teachers, Aukland, New Zealand.

Powney, J. and Watts, M. (1987). *Interviewing in educational research*. London: Routledge and Kegan Paul.

Ad Hoc Session 6

A CONJECTURE ON THE HISTORY OF MATHEMATICAL WORD PROBLEMS: WERE WORD PROBLEMS EVER PRACTICAL?

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> *I found a stone, (but) did not weigh it; (after) I subtracted one-sixth (and) added one-third of one-eighth, I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? BC 4652, 21, (Fauvel & Gray , 1987, p. 26)*
>
> *Three sailors and their pet monkey are shipwrecked on an island. they spend all day gathering a pile of coconuts and decide to divide them in the morning. But in the night, one sailor awakes and decides to take his third. He divides the pile into three equal parts, but there is one coconut extra, which he gives to the monkey. He then takes and hides his third and puts the rest back in a pile. then another wakes up and does exactly the same, and then the third. in the morning, they divide the pile that remains into three equal parts, again finding one extra coconut, which they give to the monkey. How many coconuts were there at the start? (from Ganita-sara-sangrata by Mahavira, 850 AD, in Olivastro, 1993, p. 180)*
>
> *Two men starting from the same point begin walking in different directions. Their rates of travel are in the ratio 7:3. The slower man walks toward the east. His faster companion walks to the south 10 pu and then turns toward the northeast and proceeds until both men meet. How many pu did each man walk? (Chiu-chang suan-shu, in Swetz & Kao, 1977, p. 45)*
>
> *A carpenter has undertaken to build a house in 20 days. He takes on another man and says: "If we build the house together, we can accomplish the work in 8 days." Required is to know how long it would take this other man to build it alone. (A Renaissance problem quoted in Swetz, 1987, p. 162)*

Curriculum writers sometimes justify the inclusion of word problems in school mathematics courses by saying that these problems are practical applications of mathematics to the situations of everyday life. Yet a closer examination of word problems and the solution of "real" problems in everyday life shows that the two are often very far apart (see, for example, Lave, 1992; Gerofsky, 1996). In this paper I will look at word problems dating back as far as ancient Babylonia and Egypt and ask whether word problems were *ever* practical problems. Related to this question, I will make a conjecture about the pedagogic purposes of word problems, and whether such purposes have changed historically.

Mathematical word problems have a very long history, perhaps as long as the history of human written records. Many of the mathematical cuneiform writings discovered earlier in this century on 4,000-year-old clay tablets from ancient Babylon consist of solved word problems, as do the 3,500 year old writings on the Rhind papyrus from ancient Egypt. We have recorded examples of mathematical word problems from Ch'in Dynasty China (c. 300 BC), ancient and medieval India, and medieval Europe and the Islamic world, and a continuous record of word problems from early Renaissance Europe to present-day textbooks used in school mathematics around the world.

Some of the word problems still in circulation have been shown to be identical in their mathematical structures, and often similar in their story "dressing" to problems that are thousands of years old. No one has set out deliberately to preserve these problems, yet while many other mathematical and literary forms have been lost or discarded over time, these word problems have persisted. Why?

BABYLONIAN WORD PROBLEMS: WERE THEY PRACTICAL?

> As a sixth-form student, I once got into an argument about applied mathematics at a university interview. Although I found the subject relatively easy, I did not like the way that problems were superficially about the 'real world', but in fact were so contrived that they were meaningless. The interviewer, who of course turned out to be an applied mathematician, responded rather indignantly that if the problems were not contrived they would be impossibly difficult (for mere students) to solve. This was in the early nineteen-seventies, long before the notion of modelling became fashionable in school or university mathematics. Neither I nor the interviewer had any vocabulary available to bridge the gap between our differing perspectives.
>
> \- Mathematics educator Janet Ainley on word problems as 'real' or 'unreal', a debate echoed in discussions of 'pure' versus 'applied' Babylonian mathematics.
> (Ainley, 1996, p. 1)

Hundreds of clay tablets containing "problem texts" from ancient Babylon have been found by archaeologists, most of them dating from the period 2000 to 1600 B.C. These problem texts are the second most frequently found text type on Babylonian tablets, after "table texts" containing multiplication, reciprocal, square root and other tables presumed to have been used as references by scribes. Another frequently found text type is "teachers' lists", which list alternative number sets which give integral answers to particular problem types. These are hypothesized to have been used by teachers in composing new word problems in standard forms.

The problem texts include word problems, often many on the same topic, along with instructions for solution, the answer and/or a diagram. They are believed to be textbooks from Babylonian scribal schools, which trained young people in literacy, bookkeeping and a variety of administrative duties. At first glance, the word problems appear to deal exclusively with practical problems of agricultural, commercial, legal and military administration—questions about grain stores, irrigation, inheritance, the construction of buildings and siege ramps. On account of the superficial "everyday" quality to the stories, and the fact that the scribal schools were vocational training institutions, Babylonian mathematics has been characterized as "merely practical" as opposed to later Greek abstract, theoretical mathematics.

Some scholars who have looked more closely at the corpus of Babylonian word problem texts have come to different conclusions. The problem of deixis or reference in contemporary word problems appears to originate in the very earliest word problems, those from ancient Babylon. While some of the problems included in the problem texts could conceivably refer to practical situations encountered in the day-to-day working life of a Babylonian scribe, others are very far-fetched in terms of the numbers and dimensions used, the extreme simplification of a potential practical problem, or the nature of the unknown elements and the question posed. The impractical nature of these stories calls into question their referentiality in pointing to "real world" situations, and casts doubt upon the serious practicality of even those more plausible problems.

Eleanor Robson, an Oxford Assyriologist with a mathematical background, writes,

> Should we think of [Babylonian mathematics] merely as a practical training for future overseers, accountants and surveyors?...Take the topic of grain-piles as a starting point. In the

> first sixteen problems of BM 96954 [a Babylonian mathematical tablet in the British Museum] the measurements of the grain-pile remain the same, while each parameter is calculated in turn... The first preserved problem concerns finding the volume of the top half of the pile. One could imagine how such techniques might be useful to a surveyor making the first estimate of the capacity of a grain-pile after harvest. However, then things start to get complicated. The remaining problems give data such as the sum of the length and the top, or the difference between the length and the thickness, or even the statement that the width is equal to half of the length plus 1. It is hardly likely that an agricultural overseer would ever find himself needing to solve this sort of a problem in the course of a working day.
>
> [In two other sets of problems] the pile is 10 nindan (60m) long and 36 - 48 cubits (18 - 24 m) high. It is difficult to imagine how a grain pile this big could ever be constructed, let alone measured with a stick. (Robson, 1996)

Besides grain-piles as large as an eight-storey office building and clearly impractical calculations involving unexpected combinations of their dimensions (recalling the "guess my age" word problems whose applicability continues to perplex many practical-minded students), Robson cites a Babylonian preference for integral measurements, even when it was clearly known that such measurements were inaccurate. For example, she refers to eight problems about right-angled triangles, which demonstrate three methods for finding the length of the diagonal of a 2- by 8-unit rectangle. Each of the three methods produces a different length for the diagonal, and when the length of the diagonal is given at the start of the problem, it is not the most accurate measurement which is given, but the one which will give an integer answer.

Robson argues that it is inappropriate to draw a dichotomy between "pure" and "applied" or "practical" Old Babylonian mathematics since the problems functioned on two levels. On one hand, they taught practical skills and tested methods to future scribes; on the other hand, "many of those methods no longer had real-life applications", and the problems extended originally-practical skills to story situations that were clearly not referring to everyday life.

Robson's concept of a Babylonian mathematics which was at once potentially useful and obviously impractical is appealing, and can be used to analyze contemporary mathematics. Her deliberately ambiguous view of ancient mathematics offers a way out of our current dualistic "pure/ applied" categories which seem unsatisfactory ways to describe the work of most mathematicians. In looking at the history of word problems, it is also interesting that the question, "Are they real-life problems ?" can be extended back to the earliest examples of the genre, and that ambiguous answers to that question can be traced in a continuous line back to the origins of written mathematics.

Jens Høyrup, a Danish Assyriologist and historian of mathematics, refers to a Babylonian problem about the construction of a siege-ramp in his argument about the non-applied nature of some Babylonian problems. Although the problem appears on the surface to have a practical military application, Høyrup notes that the problem solver is supposed to be able to know the "amount of earth required for its construction together with the length and height of the portion already built, but not the total length and height to be attained." He comments that "many [second-degree problems in Babylonian problem texts] look like real-world problems at first; but as soon as you analyze the structure of known versus unknown quantities, the complete artificiality of the problems is revealed...As scribal discourse in general, mathematical discourse has been disconnected from immediate practice; it has achieved a certain autonomy." (Høyrup, 1994, p. 7)

Høyrup characterizes Old Babylonian mathematics as based on *methods* where Greek mathematics grew out of problems—and this despite the fact that nearly all the Babylonian texts we have are problem texts, which delineate methods only through repeated solved examples. Høyrup's distinction is between Babylonian scribal school mathematics, which aimed to train students in methods available at hand rather than in an understanding of these methods, and Greek mathematics, which aimed to solve problems (like doubling the cube, trisecting the angle and squaring the circle) by extending mathematics and devising new methods. He comments that many of the "useless second-degree problems" included in the Babylonian texts "appear to have been chosen not because of any inherent interest but just because they *could be solved* by the methods at hand." (Høyrup, 1994, p. 7)

This analysis appears to use an anachronistic application of contemporary valuations of "training" and "education" (i.e., skills training as inferior to educated understanding). But I think that Høyrup's point is a more subtle distinction in terms of discourses available to the Babylonians and the Greeks. It is important to remember that Babylonian documents predate Greek ones by some 1500 years, and that the Babylonians may have been the first people in the world to conceive of mathematics as a unified and distinct area of study. Høyrup writes,

> There were no social sources, and no earlier traditions, from which a concept of mathematics as an activity per se could spring, and there was thus no possibility that a scribe could come to think of himself as *a virtuoso mathematician.* Only the option to become a *virtuoso calculator* was open; so, Babylonian "pure mathematics" was in fact calculation pursued as *art pour l'art*, mathematics applied in its form but disengaged from real application. (Høyrup, 1994, p. 8)

He does not imply that the Babylonians were incapable of "pure mathematics", but that they had no other discourse available but that of practical problems to express abstract mathematical ideas. "Even when Old Babylonian mathematics is 'pure' *in substance*, it remains *applied in form.*," writes Høyrup. " In contradistinction to this, the prototype of Greek mathematics is pure in form as well as in substance."

On the other hand, Jacob Klein, writing about Diophantus, reserves the notion of "purity of form" for modern symbolic algebra. For him, Diophantus' mathematics was an intermediate stage bridging the "rhetorical algebra" of problems stated in words and ordinary language, and the abstract symbolic algebra of modern mathematics:

> We must not forget that *all* the signs which Diophantus uses are merely word abbreviations... For this reason Nesselmann (*Algebra der Griechen*, p. 302) called the procedure practiced by Diophantus a "syncopated algebra" which, he said, forms the transition from the early "rhetorical" to the modern "symbolic" algebra (according to Nesselmann even Vieta's mode of calculation belongs to the stage of syncopated algebra). (Klein, 1968, p. 146)

Again the questions raised in discussion of Babylonian word problems can be viewed in a way that reflects on our use of word problems in contemporary school mathematics. Are word problems used primarily to train students in the use of methods, without necessarily understanding those methods? Are problems chosen simply to illustrate the "methods at hand"?

Similarly, we could ask whether our need for the concrete imagery of word problems has changed, since we have access to post-Greek abstract mathematical discourse and symbolic algebra.

THE RELATIONSHIP BETWEEN THE HISTORY OF WORD PROBLEMS AND RIDDLES, PUZZLES AND RECREATIONAL MATHEMATICS

I have argued elsewhere (Gerofsky, 1996) that mathematical word problems are generically non-referential and that it is inappropriate to assign truth value to them—they flout the Gricean maxim of quality. That is to say, even when word problem stories appear to refer to aspects of the "real world", their links to the world of lived experience are ambiguous at best. So why are these rather fanciful stories included at all?

I raised this question in correspondence with mathematician David Singmaster of South Bank University, London. Singmaster, who has done extensive research into the history of recreational mathematics, replied:

> Many problems in recreational mathematics are embellished with a story which is often highly improbable and this is partly what makes the problem memorable and recreational. However, I don't know which came first, the problem or the story. In many cases, the story is essential to make the problem interesting. E.g., I once saw an exam schedule where 6 2 1/2 hour exams was typed as 62 1/2 hour exams. The first takes 15 hours and the second takes 31 hours. When can they be equal? This formulation is much more interesting than, "Find numbers a, b, c, d such that a * (b + c/d) = ab * c/d". (Singmaster, 1996)

Singmaster's comments raised several issues, including the question of the sheer pleasure of story, and particularly nonsensical story, in word problems, and the historic and generic relationship between word problems in schools and orally-transmitted riddles in social settings.

Høyrup has addressed both these questions to some degree in his writing on Babylonian word problems. He quotes Hermelink (1978, p. 44) describing recreational mathematics as "problems and riddles which use the language of everyday but do not much care for the circumstances of reality". "'Lack of care' is an understatement," writes Høyrup, and echoes Singmaster in saying that "a funny, striking, or even absurd deviation from the circumstances of reality is *an essential feature* of any recreational problem. It is this deviation from the habitual that causes amazement, and which thus imparts upon the problem its recreational value. " (Høyrup, 1994, pp. 27- 29)

Using "the language of everyday" but "not much caring for the circumstances of reality" is also a very apt characterization of the non-referential nature of both word problems and parables as genres. As shown above, a "lack of care for the circumstances of reality" has been a feature of word problems as early as the Old Babylonian period. Høyrup posits a continuum of non-referential story problems ranging from the most delightful mathematical recreations to the dullest of school exercises:

> One function of recreational mathematics is that of teaching... This end of the spectrum of recreational mathematics passes imperceptibly into general school mathematics, which in the Bronze Age as now would often be unrealistic in the precision and magnitude of numbers without being funny in any way. Whether funny or not, such problems would be determined *from the methods to be trained...* Over the whole range from school mathematics to mathematical riddles, the methods or techniques are thus the basic determinants of development, and problems are constructed that permit one to bring the methods at hand into play. (Høyrup, 1994, pp. 27- 29)

Work by the German mathematics historian Tropfke (1979), Singmaster (1988) and others has established the often ancient provenance and wide historical, geographic and cultural distribution of a large number of famous word problem/ recreational problem types. For example, the "cistern problem", the "purchasing a horse" problem, the "100 birds" problem, and the "crossing a river" problem have appeared in ancient India and China, medieval Byzantium and the Islamic world, and in medieval and Renaissance Europe. The familiar children's riddle in English, "As I was going to St. Ives," has been traced back to a problem in the Rhind papyrus and is related to a problem from Sun Tzu in ancient China; and the famous Islamic-Indian "chessboard problem" can be traced to Babylonian origins, as well as related problems in China and in Europe (in Alcuin, for example). As Høyrup points out, "the same stock [of widely-known story problems] was also drawn upon by Diophantus, who, of course, stripped the problems of their concrete dressing."(Høyrup, 1994, p. 35)

Høyrup relates the very wide distribution and longevity of these famous problems to an oral tradition of recreational problem riddles transmitted by merchants along the Silk Route:

> Like other riddles, recreational mathematics belongs to the domain of oral literature. Recreational problems can thus be compared to folktales. The distribution of the "Silk Route group" of problems is also fairly similar to the distribution of the "Eurasian folktale", which extends "from Ireland to India"... However, for several reasons (not least because the outer limits of the geographical range do not coincide) we should not make too much of this parallel. Recreational problems belong to a specific subculture -- the subculture of those people who are able to grasp them. The most mobile members of this group were, of course, the merchants, who moved relatively freely or had contacts even where communication was otherwise scarce (mathematical problems appear to have diffused into China well before Buddhism). (Høyrup, 1994, pp. 34-35)

The notion that recreational mathematics problems were orally transmitted across Europe and Asia is supported by observations from other writers on the history of mathematics. Dominic Olivastro cites the following examples of problems from Alcuin:

> A certain gentleman ordered that ninety measures of grain were to be moved from one of his houses to another, thirty half-leagues away. One camel was to transport the grain in three journeys, carrying thirty measures on each journey. A camel eats one measure each half-league. How many measures will be left, when all has been transported? (Alcuin of York, in Olivastro, 1993, p. 131)
>
> A merchant in the East wanted to buy a hundred animals for a hundred shillings. He ordered his servant to pay five shillings for a camel, one shilling for an ass, and to buy twenty sheep for a shilling. How many camels, asses, and sheep were involved in the deal? (Alcuin in Olivastro, 1993, p. 133)

...and comments:

> Camels appear often enough in these puzzles to raise a few suspicions...Why would an English monk [Alcuin] frame his puzzles in terms of an animal he probably never saw? The answer is that the puzzle, like many in the *Propositiones*, originated in the Middle East. The Arabs had already learned of the positional number system from the Indians, who in turn may have received it from the Chinese. The puzzles are from a very talented people; the solutions [that Alcuin gives] are not...[Alcuin] gives the wrong answers, or misunderstands the problems, or fails to find the general principle behind them. (Olivastro, 1993, p. 133)

Olivastro also quotes the following variants on the "crossing the river" problem from Alcuin:

> Three men, each with a sister, needed to cross a river. Each one of them coveted the sister of another. At the river, they found only a small boat, in which only two of them could cross at a time. How did they cross the river, without any of the women being defiled by the men? (Alcuin, *Propositiones* problem 17, in Olivastro, 1993, p. 136)

> A man had to take a wolf, a goat, and a cabbage across the river. The only boat he could find could take only two of them at a time. But he had been ordered to transfer all of these to the other side in good condition. How could this be done? (Alcuin, *Propositiones* problem 18, in Olivastro, 1993, p. 138)

…and reports the following, unattributed, anthropological lore:

> In the Swahili tradition, a visitor from another region visits a sultan but refuses to pay tribute. He is confronted with a challenge: He must carry a leopard, a goat, and some tree leaves to the sultan's son who lives across a river, and he must use a boat that will hold only the visitor and two other items. The problem, of course, is that no two items can be left on the shore together…The visitor, after mulling over the problem, decides to carry first the leaves and goat, return with the goat, and then carry the goat and the leopard together to the son.
>
> A similar idea is found in Zambia. This time there are four items to transport: a leopard, a goat, a rat, and a basket of corn, where each is likely to eat the one following it. The boat can hold only the man and one item. The story tells us that the man considers leaving behind the rat or the leopard, and thus reducing the problem to the one of the Swahili tradition, but, the story goes, the man finally realizes that all animals are his brothers—so he decides not to make the trip at all! (Olivastro, 1993, p. 139)

I would have viewed these unattributed "exotic tales" with some degree of skepticism had it not been for a similar report, this time from the Atlas mountains of Morocco, from Eric Muller in a recent CMESG newsletter (Muller, 1994). Muller visited several remote villages in the Atlas mountains on a number of occasions, accompanied by a Moroccan colleague, Professor Ha Oudadess of Rabat. In 1979 Professor Oudadess had carried out a survey of oral mathematics traditions in the Atlas, and was able to translate conversations from Berber into French for Dr. Muller.

Here is Eric Muller's account of his experience of a living oral tradition of mathematical problem solving:

> It was in [the isolated village of] Tizi n'Isli, on the terrace of the only cafe in town, over the inevitable cup of sweet mint tea, that we talked to Qlla Ikhlef. He speaks only Berber, which [Professor] Ha [Oudadess] translates for me into French… He was born in 1962, in a small village higher up in the mountains…He has no formal education but recalls that at the age of fourteen he was allowed to sit with some elders who would spend time inventing and solving problems. Can he recall any of the problems? "Oh yes."
>
> "Three people own 30 she-goats. In the spring 10 of them bear 3 kids, 10 bear 2 kids and 10 bear 1 kid. The three owners decide to split the herd equally so that they have the same number of goats and kids. No kid is to be separated from its mother. How can we do this?"…
>
> Qlla recalls a social 'problem competition' between men from his village and others from a different village. This would be held at the time of the souk (the travelling market which, in small villages, is held once a week.) On warm nights, before the souk, men would gather in an open tent, drink sweet mint tea, talk and sometimes pose problems for members from the other village to answer. We wonder where this tradition comes from. What in this culture makes mathematics problems solving a social activity? We know it is quite old because Ha remembers that as a youngster, in the Atlas, he was stopped and challenged by elderly men from the village to solve problems. It is also quite extensively spread throughout the Moroccan Atlas…

The word has spread that these strangers, in shorts, are interested in problems. A group of men come forward, one of them wants to share a problem. It is a version of the river-crossing involving a boat, a wolf, a goat and a cabbage, reworded for the local situation. No one else in the crowd has a problem to share....

Here are a couple more problems gathered by Ha [elsewhere in the Atlas]:

1) A person wishes to purchase 100 birds using exactly 20 rials. For one rial the person can get either one hawk, or two pigeons or five sparrows. The person must buy at least one bird of each type. How many of each type can the person buy?

2) Three people go to the barber. To pay each person opens a drawer and places a payment equal to what is found in it and then closes the drawer. After the customers have left the barber finds 10 rials in the drawer. He wants to know how much each person paid. (Muller, 1994)

A CONJECTURE ABOUT THE PEDAGOGIC PURPOSES OF WORD PROBLEMS

Looking at the examples above, it seems clear that mathematical word problems, from their very inception, have not simply been applications of mathematics to real-life problem situations. Rather, word problems as a genre are complexly and ambiguously related to parables, riddles and folktales. They also seem to have served the purpose of "bodying forth" particular mathematical ideas and methods in the guise of stories about purportedly real things, although their contact with the world of mathematical ideas often seems to be closer than their contact with the "things" of our lived experiences.

My conjecture is that the need for such stories has changed historically, particularly with the introduction of algebra, and that the current emphasis on the purported practicality of word problems developed in reaction to their change of purpose.

In a pre-algebraic world, there is no explicit and compact way to state many mathematical ideas in a general form. The most convenient and effective way to convey the idea of a mathematical generalization, particularly in a pedagogic setting, may have been by heaping up repeated examples of a particular concept or method, in the form of numerous, slightly differing stories. This is in fact what we find in mathematics teaching texts, from Babylonian times onward. I speculate that mathematics instructors from pre-algebraic times must have hoped that their students would eventually be able to "see through" the language of the stories to the mathematical generalizations that they pointed to. (And in this discussion, it is important to remember that even now, students up to the senior years of secondary school may be considered "pre-algebraic"!)

With the introduction of algebra to western Europe in the early Renaissance, it became possible to state mathematical relationships in a concise form which did not depend on story or example. Coincidentally, justifications of word problems as examples of practical, useful problem solving for the world of work began to appear in many Renaissance texts, and have continued to feature prominently in curriculum writing since then.

My conjecture is that the pedagogic purposes for word problems changed with the introduction of algebra. Before algebra, word problems were inseparable from mathematics because they provided the only way of talking about many mathematical ideas—that is, by talking "through" a parabolic example. With the advent of algebra, word problems became detached from this formerly self-evident purpose. Algebra provided a more direct, elegant means of expressing mathematical generality, yet the word problem genre continued to appeal to mathematics teachers. Perhaps in lieu of other explanations, the pretext of word problems as practical, useful applied mathematics arose to justify the continued use of word problems in mathematics teaching.

And why have word problems continued to appeal to teachers, even if one of their principal original purposes disappeared with the introduction of algebra? Perhaps mathematics teaching is simply a tremendously conservative enterprise. Perhaps there is a sense of a link with a long, unbroken tradition which extends across four millennia and many widely-separated cultures. Perhaps word problems provide a connection, though sometimes a tenuous one, between schooling and the pleasurable worlds of riddle, puzzle and game. Perhaps there is a basic human need to clothe abstractions in the guise of story, a need as familiar and strange as the world of our own dreams…

I would be very interested in hearing from anyone whose research might shed light on these conjectures!

REFERENCES

Ainley, J. (1996). *Constructing purpose in mathematics classrooms*. Unpublished doctoral dissertation, University of Warwick, England.

Fauvel, J. and Gray, J. (Eds.) (1987). *The history of mathematics: A reader*. London: Macmillan.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics*, 16 (2).

Hermelink, H. (1978). Arabic recreational mathematics as a mirror of age-old cultural relations between Eastern and Western civilizations. In Hassan, A. Y. , Karmi, G. and Namnum, N. (Eds.), *Proceedings of the first international symposium for the history of Arabic science*, April 5 - 12, 1976. Vol. 2, Papers in European languages. Aleppo, Syria: Institute for the History of Arabic Science, Aleppo University.

Høyrup, J. (1994). *In measure, number, and weight: Studies in mathematics and culture*. Albany, N.Y.: State University of New York Press.

Klein, J. (1968). *Greek mathematical thought and the origin of algebra*. New York: Dover.

Lave, J. (1992). Word Problems: A Microcosm of Theories of Learning. In Paul Light and George Butterworth (Eds.), *Context and Cognition: Ways of Learning and Knowing* (pp. 74 - 92). New York: Harvester Wheatsheaf.

Muller, E. (1994, December). Morocco for ICOTS 4. In Newsletter of the Canadian Mathematics Education Study Group. (Available from Dr. David Reid, Memorial University, St. John's, Newfoundland).

Olivastro, D. (1993). *Ancient Puzzles*. New York: Bantam.

Robson, E. (1996). Personal communication.

Singmaster, D. (1988). Some early sources in recreational mathematics. In Hay, C. (Ed.), *Mathematics from manuscript to print 1300 - 1600* (pp. 195 - 208). Oxford: Clarendon Press.

Singmaster, D. (1996). Personal communication.

Swetz, F. J. and Kao, T. I. (1977). *Was Pythagoras Chinese*? Reston, VA: National Council of Teachers of Mathematics.

Swetz, F. J. (1987). *Capitalism and arithmetic*. La Salle, Ill.: Open Court.

Tropfke, J. (1980). *Geschichte de Elementarmathematik, 4. Auflage: Band 1: Arithmetik und Algebra*. Berlin: De Gruyter.

Ad Hoc Session

DESPERATELY SEEKING SOMETHING: DILEMMAS SURROUNDING THE INTERPRETATION OF TEACHERS' INTERVENTIONS

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This paper considers the dilemmas that educational researchers face in analysing video data of classroom activities. In particular, I am concerned with dilemmas surrounding the interpretation of teachers' interventions.[1] In analysing the data I have collected during my research I have employed the Pirie-Kieren Dynamical Theory of the Growth of Mathematical Understanding as a theoretical tool. Susan Pirie and Tom Kieren have been refining their theory of the growth of mathematical understanding for many years, and have published their reflections and findings widely (Pirie and Kieren, 1989; Pirie and Kieren, 1994a; Pirie and Kieren, 1994b). It is not my intention to furnish a complete description of their theory, although I encourage an interested reader to gain a deeper understanding than I am able to offer here by reading Pirie and Kieren's own reflections. There are, however, a number of features of the theory which are pertinent to the discussion I wish to pursue, and so, although this may be repetitious for some readers who have closely followed the development of the 'Growth of Mathematical Understanding' model over the last decade, I will take a few moments to explicate the critical notions.

A MODEL OF THE GROWTH OF UNDERSTANDING

Figure 1 shows a diagrammatic representation of the Pirie-Kieren 'Growth of Mathematical Understanding' model. This representation shows the various levels or modes of understanding that Kieren and Pirie have identified. It should be stressed at this point that Pirie and Kieren have been at pains to indicate that they do not believe that understanding is a uni-directional phenomenon. Rather, Pirie and Kieren (1994a, p. 39) have observed understanding as a "whole, dynamic, levelled but non-linear, recursive process." The innermost mode of understanding has been termed *Primitive Knowing* (PK). This knowing is the history or experiences that the student brings to the situation. The second mode or level of understanding is termed *Image Making* (IM). Here the student is observed to be engaged in activities aimed at helping him or her develop particular images. The learner is being asked to make distinctions in her or his previous abilities and use them in new circumstances or to new ends. After engaging in such activities a student may be able to replace those activities with a "mental plan." This is *Image Having* (IH), which frees the student from the need for particular activities or examples. The remaining modes of understanding will not be relevant to my discussion.

A further critical feature of the model is the notion of *folding back*. The model is a set of unfolding layers, suggesting, both visually and conceptually, that each layer enfolds, unfolds from, and is connected to, each inner less formal layer. Despite the visually seductive image of "growing" rings embodied in the diagrammatic representation of the model, it is Pirie and Kieren's espoused belief that understanding does not happen that way (in other words, it is not a linear,uni-directional process). They argue that growth in

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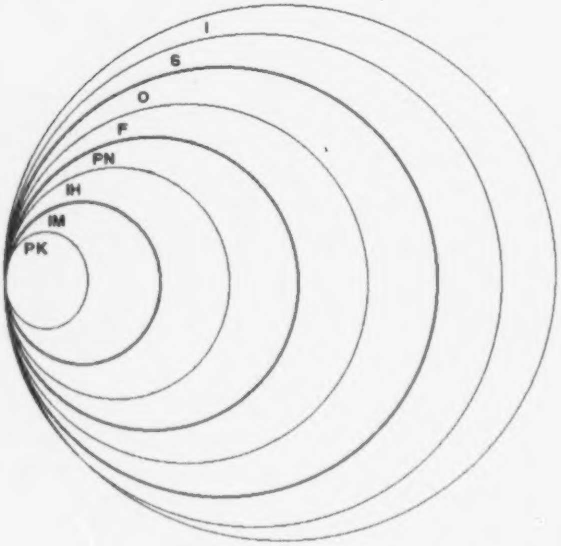
[1] I am using this term in the same manner as it has been applied by Kieren and Pirie (1992), and I will define it in a moment.



Key:

PK	Primitive Knowing
IM	Image Making
IH	Image Having
PN	Property Noticing
F	Formalising
O	Observing
S	Structuring
I	Inventising

Figure 1

understanding entails repeated *folding back* to inner, less formal understandings (Pirie and Kieren, 1994b). It should be stressed that this returned-to mode of understanding is not the same as the previous understanding observed at this level. It has now been shaped by the more sophisticated understandings generated at an outer level.

I now wish to turn to the feature of the model in which I am particularly interested, both in terms of the issues I wish to raise in this paper, and also in terms of my own research. This feature is the nature of Pirie and Kieren's treatment of the role of the teacher. Kieren and Pirie (1992) have identified three kinds of interventions: *provocative*, *invocative* and *validating*, which may be recognised in the behaviour and talk of both teachers and students. *Invocative* interventions are those which result in the student *folding back* to an inner level of mathematical understanding. *Provocative* interventions are those which result in a movement to an outer or more sophisticated level of understanding. *Validating* interventions are those which serve to check a student's understanding, and/or encourage the student to express (verbally, symbolically, etc.) their current mathematical ideas. Pirie and Kieren note that they believe that it is the response of the student, and not the intent of the teacher, which determines the nature of the intervention.

As I have begun to work closely with my video-taped data I have been faced with a number of episodes which resist this categorisation. It is one such episode that I have chosen to discuss in this paper. I intend to provide more detail about the students, the teacher, and the activity in a moment, however, I first wish to explain what interests me about the episode. The episode features a teacher working with a pair of students, one of whom is having difficulty finding the perimeter of a given shape. Though the teacher continues for some minutes to help the struggling student, it is clear by the end of the episode that the student still does not understand the concept being discussed. My interest was piqued by this episode in particular due to the dilemma I faced in trying to categorise the teacher's interventions. As I have indicated, Pirie and Kieren categorise interventions based on outcome. In this case it is possible to discern from the video that the struggling student makes little progress. In terms of the theoretical model his actions and verbalisations indicate that he is *Image Making* throughout the interaction. He appears neither to move outwards to a more sophisticated understanding, nor to *fold back* to an inner mode of understanding. As his actions and language enable us to verify his current level of understanding (*Image Making*) we are lead to the conclusion that the teacher's interventions should be labelled *validating*. It is here that I begin to be troubled by this classification. I am faced with a dilemma. Though categorising teachers' interventions by student outcome seems, in most cases, to further the analysis, I remain troubled by some episodes. A more detailed rendering of the particular episode I have mentioned may now inform the discussion.

THE TROUBLESOME EPISODE

The classroom episode I am about to describe was video-recorded as part of my on-going inquiry into the influence of teachers' interventions on the growth of students' mathematical understanding. The data from which this episode is selected were collected in my own classroom in a British high school at a time when I was a full-time teacher of mathematics, simultaneously engaged in a study of my own practice. Three pairs of Grade 7 and 8 students (corresponding to Grades 6 and 7 in North America) were video-taped during several weeks of mathematics lessons, and each student was then video-taped in a one-to-one interview with me at the end of the series of lessons. The selected episode focuses on a Grade 8 student (equivalent to Canadian Grade 7), Donny (D), and his classmate, Sula (S), and features Donny attempting to find the perimeter of the shape in Figure 2. Donny calls over the teacher (J) and asks "What do you put for this bit?" pointing to the horizontal line section with the missing value. The interaction continues:

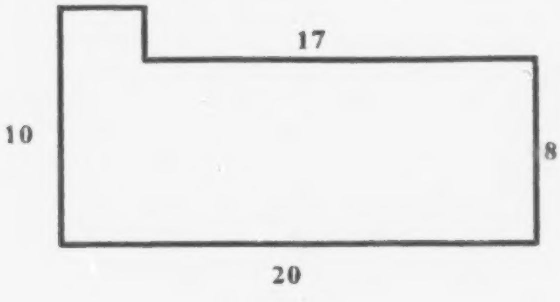


Figure 2

J: Ah, well now. We've got to work that out.
D: Do you think that's ten, or twenty you put?
J: No, it's neither of those. Tell me the whole length of the rectangle. The whole thing, of the picture, the whole length.
S: You've already got the answer down, Donny.
D: Eight?
J: No, well, no. [*Pause*] Tell me the length of the whole shape. [*Donny picks up his ruler*] No.
S: No, you don't need your ruler, Donny, you've got it written down.
J: Read it off. [*Pause*] What's this distance? [*Pointing to the bottom of the shape*]
D: Twenty centim...
J: Twenty. Twenty something. We don't know what, but twenty. OK and what's this part? [*Drawing her finger along the section marked 17*]

D: Eight.
J: No.
D: Seven…seventeen.
J: Seventeen. So what's this part? [*Pointing to the horizontal line section with the missing value*]
D: Seventeen.
J: [*Shakes head*]
D: Twenty!
J: No, the whole thing is twenty. That much is seventeen. [*Pointing to the length marked 17*]
D: Oh, thirty seven.
J: [*Shakes head*]
D: Is it ten, then?
J: No.
D: Eight?
J: No, no don't guess, don't guess.
S: Can I tell him the easy way of working it out?
J: No. I want him to get there for himself. The whole distance is twenty to there, do you agree? And the distance to there is seventeen, so how much must that be?
S: Oh…God!
J: No, it's all right, Sula. Donny'll get there. The whole distance is twenty…..

The following few minutes of video-tape show Donny desperately seeking the answer to his question, and the teacher desperately seeking to provide the answer without directly 'telling'. It is clear by the end of the extract that neither party is entirely satisfied with the outcome. Donny, I am convinced, would have appreciated a simple and straight answer to his question, as there is ample evidence later in the extract to suggest that he is able to work out the perimeter of the shape given all the dimensions. The teacher (I know, for it was me) would, at least once during the episode, have liked to give Donny the straight answer he sought. And yet the dance continued, Donny never directly re-stating his question, the teacher never directly answering it. The frustration for both (and for Sula who is desperate for Donny to grasp the connection between the three horizontal line sections) is evident on the video. My point in this paper, though, is not to explore the dilemmas for the teacher who wishes to refrain from immediately telling her students the answers to their questions, although these dilemmas are real and worthy of exploration. Instead, I aim to explore the dilemmas faced by the researcher who seeks to make sense of particular classroom interactions captured on videotape. In this particular instance I am interested in the interaction between a teacher and her students. One difficulty with categorising teachers' interventions solely by considering the subsequent actions and verbalisations of the student is that it renders the role of the teacher secondary to that of the student. I am not implying that we should treat the role of the teacher as any more significant than that of the learner, rather that in privileging one we run the risk of diminishing the other. To be true to the essence of the interaction (of which I was a part) I cannot neglect the intent of the teacher, which most definitely was not simply to encourage Donny to verbalise his current mathematical thinking (suggesting *validating* intent), but to extend his working image of measure by helping him to see the relationship between the various line sections of the shape (suggesting, perhaps, a *provocative* intent). I therefore want my theoretical mapping of the incident to reflect not only the actions and verbalisations of the learner, but also those of the teacher. Dealing with the intent of a research participant is a thorny issue, to which I will return in the next section. No matter what my intent at the time, though, the fact remains that I was not successful in my teaching effort. What I might have done differently to help Donny to understand is the subject of another paper currently in preparation[2]. Here, I wish to continue to address the issue of what, as a researcher, I might gain from a study of this episode. There are, I think, several choices before me.

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[2] I am currently collaborating with Keith Roy, who was a participant in my Ad Hoc session at Thunder Bay, on an article devoted to this issue.

CHOICES AND DILEMMAS

In research we are always faced with choices, and what we choose is ultimately reflected in the data we obtain, and the findings we generate. My research is no different in this respect from any other study, however, the choices researchers make, and the struggles they have with their data are not always explicitly formulated in academic articles. Increasingly this bias is being corrected. As long ago as 1966, in his Nobel Lecture, Richard Feynman commented on this very issue:

> We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work (quoted in Mackay, 1991, p. 91).

Fortunately, thanks in part to organisations such as CMESG/GCEDM, we now have the opportunity to think-out-loud, to work with our peers on a stimulating but thorny issue and later to publish, in a dignified manner, the results of those deliberations. I have chosen to discuss my struggles here because I believe that my reflections might resonate with others and open an interesting discussion of possibilities, not only for myself, but for others present at the Ad Hoc session or reading this paper later in the year. It seems that there are a number of possibilities open to me for proceeding with categorising the chosen episode.

I could simply label this episode, and other similar ones, as 'resisting categorisation', and, at least for the moment, leave it at that. However, I am intensely curious about this episode and to reject it, even temporarily, from the data simply because it is troublesome seems not only cowardly, but also foolhardy. My gut reaction is that it is just such an episode which strikes at the heart of my area of interest (the role of teachers' interventions in influencing the growth of students' mathematical understanding). So, for the moment, I am reluctant to bypass this episode. I have shown this video extract to numerous groups of pre-service and practising teachers, and to teacher educators and fellow researchers. Their overwhelming impression has been that the teacher is clearly trying desperately to teach, but that Donny does not seem to learn. As a teacher I am fascinated by such episodes, and believe that they can speak volumes about the role of the teacher in influencing the growth of understanding. As a researcher I am struggling to articulate those beliefs.

So, if I am not willing to abandon this episode, I need to be comfortable with the categorisation I make concerning it. Because I know what I intended as the teacher in this situation (which was to encourage Donny to move to a more sophisticated level of understanding, in other words, to encourage him to "see" the connection between the various horizontal line sections of the diagram) I could propose that my actions be labelled as an "unsuccessful *provocative* intervention". From my researcher-perspective this produces a further dilemma, however, concerning the amount and type of data which must be collected. Each time I view videotape recorded in my own classroom I am, as a researcher, in the privileged position of knowing what was intended by the teacher featured on the video each time she made an intervention. However, when the video features a teacher other than myself I am immediately at a loss (without resorting to extensive video-stimulated recall sessions) to know the teacher's intention. With each step, then, the dilemmas seem to grow. At the heart of each dilemma is a research question

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worthy of serious consideration, such as 'Can we infer intent?' and 'What kind of evidence would enable us to do so?' Such questions raise further and deeper questions about the research process, such as 'What is the difference between an inference from the data and an interpretation of those data?' and 'Is one any more valid than the other?' I am still wondering about answers to these questions.

For the moment I will go no further with my deliberations, but leave it to the reader to reflect on the appropriateness and consequences of the choices I have already made and am still considering. Whilst for me this dilemma remains unresolved for the moment, I hope that in deliberating on paper in this way I have reminded my readers of the value of reflecting aloud, and enabled readers to explore the varied possibilities for dealing with dilemmas in their own work. I conclude with thanks to Donny for his patience when mine was running out, and with a favourite quote from poet Rainer Maria Rilke which I mentally invoke whenever the dilemmas of research seem to be mounting:

> *Be patient toward all that is unsolved*
> *in your heart*
> *Try to love the questions themselves.*

REFERENCES

Kieren, T. and Pirie, S. (1992) The Answer Determines the Question: Interventions and the Growth of Mathematical Understanding, *Proceedings of the 16th Annual Meeting of the International Group for the Psychology of Mathematics Education*, vol II, New Hampshire, 1-8.

Mackay, A. (1991) *A dictionary of scientific quotations*, Bristol: Adam Hilger.

Pirie, S. and Kieren, T. (1989) A Recursive Theory of Mathematical Understanding, *For the Learning of Mathematics*, 9, 3, 7-11.

Pirie, S. and Kieren, T. (1994a) Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments, *For the Learning of Mathematics*, 14, 1, 39-43.

Pirie, S. and Kieren, T. (1994b) Growth in Mathematical Understanding: How Can We Characterise It and How Can We Represent It?, *Educational Studies in Mathematics*, 26, 2/3, 165-190.

Ad Hoc Session 8

SCARBOROUGH REVIEW OF GRADE 12 MATHEMATICS

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INTRODUCTION

The Scarborough Board of Education is committed to review key curricular areas at the end of each division on a four to five-year cycle for the purposes of accountability and program improvement to enhance student learning. The Scarborough Reviews are designed to look at the entire program at a grade level in a subject area and a wide range of student skills and abilities. All students registered in a Grade 12 Mathematics course in all semestered and full year secondary schools in Scarborough participated in the Mathematics Review in April-June, 1996 and all Semestered 1 students registered in a Grade 12 Mathematics course in all semestered secondary schools participated in the review in November, 1996 to January, 1997.

THE MATHEMATICS CURRICULUM

The 1996/97 Scarborough Grade 12 Mathematics Review tests were based on the *Ontario Secondary/Intermediate School Guidelines for Mathematics* (OS:IS) (1985). The MAT4A courses are further elaborated and supported by *the MAT4A course for Scarborough (1992).* The MTB4G courses are supported by the *Mathematics for Business and Consumers, Grade 12 General Level* while the MTT4G course is further elaborated by the Scarborough Curriculum document written in 1989, *Mathematics for Technology (Applied Mathematics), MTT4G, Grade 12.* The curriculum and assessment standards fostered in the *NCTM Curriculum and Evaluation Standards for School Mathematics (1989)* are also incorporated in the assessment activities of the teaching units.

The content and topics included in the Review were selected from course outlines submitted by teachers teaching the relevant grade 12 courses to ensure they were core content areas in the *OS:IS Guidelines* and the mathematics curriculum in most Scarborough schools. The areas tested were:

- **MAT4A**
  - Algebra
  - Logarithms and Exponents
  - Algebra of Functions
  - Trigonometric Functions
  - Cumulative Numeracy
- **MTT4G/MTB4G**
  - Numerical and Algebraic Methods (including Cumulative Numeracy)
  - Geometry
  - Trigonometry

- Forces and Vectors

- **MTL4B/MTW4B**
  - General Numeracy
  - You and Your Money
  - Math on the Job
  - A Place of Your Own

THE MATHEMATICS REVIEW INSTRUMENTS

The Scarborough Grade 12 Mathematics Review used several distinct instruments to generate a description of the Grade 12 Mathematics program in Scarborough. The instruments included:

- a School Questionnaire which provided a profile of the total school environment and instructional philosophy for mathematics;
- a Teacher Questionnaire which provided information about the teachers' backgrounds and the teaching/learning strategies used in delivering the mathematics program;
- a Student Questionnaire which provided information about students' attitudes towards school, as well as the mathematics program;
- a Teacher Opportunity-to-Learn Form which provided information about the extent to which the mathematics program was being taught.

Student achievement was evaluated through a 2-week assessment unit and a paper-and-pencil test. Although every student in Grade 12 participated in the assessment, there are no scores for individual students. The review used a multi-matrix sampling approach whereby different groups of students responded to different sets of items and projects. Although teachers were asked to use information from the project and paper-and-pencil assessment to allocate 10% of the term mark for each student, the emphasis was on providing a system-wide portrait of the performance of the group of students in Grade 12 in relation to the expected outcomes.

Teaching Unit

Three model assessment units were designed to give students direct hands-on instruction in the selected topics, through specific activities and assignments. The units were developed using an integrated approach to learning to make mathematics more concrete and relevant for students.

- MAT4A: the focus for the advanced level unit was on *Mathematical Modelling*. Students analyzed the relationship between two variables using exponential or power functions to model some phenomenon, and then applied the derived model to elicit appropriate information and make relevant predictions for future events based on the model.

- MTT4G/MTB4G: the focus for the general level unit was on the *Stock Market*. Students actively tracked stock prices using a personally-selected portfolio to learn the basics of investment.

- MTL4B/MTW4B: the focus for the basic level unit was on *Percentages and Interest*. Students were given step-by-step instruction in the application of percentages and interest using money. Being able to use money correctly gives these students a practical advantage in understanding consumer mathematics.

Paper-and-Pencil Test

The paper-and-pencil component of the review included a test comprised of a variety of question types, e.g., multiple-choice questions, numerical problems, and problems that required full written solutions. Also, included in the paper-and-pencil booklets were questions that required students to create charts, tables, and graphs from given data and/or formula, and that required them to interpret data from given charts, tables, and graphs to solve problems.

THE SCORING OF STUDENT WORK

The Scoring Process

The scoring of the work of the 1995/96 cohort was conducted centrally at the Scarborough Board offices over a 3-week period in July, 1996. The marking of the 1996/97 semester 1 students' work was conducted in December, 1996 and in Jan. 1997. The projects and all problem-solving questions on the paper-and-pencil booklets were scored by teacher-markers. Simple responses from the paper-and-pencil instruments were coded by clerical staff or student markers. Markers were grouped into teams and worked under the direction of trained group leaders. The group leaders were trained by the staff of the Mathematics Department prior to the July scoring period. Twenty teachers from the secondary panel participated in the scoring process. Several checks were built into the process to ensure reliability and accuracy.

Scoring Scales

Student achievement in the *teaching unit* was measured against holistic scoring scales developed by Scarborough educators for each outcome category. The development of the scoring scales was based on the model developed by the Ministry of Education for the *Grade 12 Provincial Writing* and *Grade 9 Provincial Reading and Writing Reviews*, the model developed for *the Scarborough Review of Literacy, Primary and Junior Divisions, 1993-94* and *the Scarborough Review of Science, Intermediate Division 1994-95*. The Grade 12 Mathematics outcomes were regrouped into four reporting outcome categories. The projects were evaluated four times, once for each of the outcome categories.

Outcome Categories	Scoring Scale
Communication	three-level scale
Technology/Mathematics Conventions	three-level scale
Mathematics Applications	four-level scale
Overall Mathematics Performance	six-level scale

The holistic scales that describe levels of performance in the four reporting outcome categories are outlined on the following pages. The strongest level in each category was assigned the highest level, and the weakest was assigned level 1. The levels include descriptions of what the respective categories "look like". They should not be equated with marks, percentages, grades, or pass/fail. *The student work was scored according to these descriptive levels of achievement, and not in comparison to other students.*

Each question in the *paper-and-pencil instruments* was assigned a level depending on the skills and thinking required to arrive at the "best response". The highest possible level was assigned to the "best response" while partial responses were also recorded. Most of the Short Answer questions had an obvious "best answer" and were scored according to this method. In addition, common errors that were made by students were analysed and coded.

SCORING SCALES FOR THE FOUR OUTCOME CATEGORIES FOR MAT4A TEACHING UNIT

Outcome Categories	*Level 1*	*Level 2*	*Level 3*
Communication	The performance demonstrates a fragmented organization of the project which prevents the communication of the investigation to the readers. Data, results and conclusions are presented with minimal or no support.	The communication demonstrates adequate organization of the project to communicate most of the elements of the investigation to the readers. Data, results and conclusions are presented with limited support.	The performance demonstrates an effective organization of the project to communicate the investigation clearly to the readers. Data, results and conclusions are clearly presented and supported
Technology	The performance demonstrates minimal command of the technology. The report of the project are prepared using the minimal capabilities of the software.	The performance demonstrates adequate command of the technology. Parts of the report of the project are prepared using some fundamental capabilities of the software.	The performance demonstrates a good command of the technology. The report of the project are effectively prepared using the full capabilities of the software.
Application and Comprehension	Student demonstrates minimal grasp of the very basic mathematical concepts and little evidence of applying them to other situations. Most of the steps of modelling are unstructured, incomplete and solutions to investigations are not understandable or omitted.	Student demonstrates partial understanding of some basic mathematical concepts of modelling and often unable to apply them to new situations. Some basic steps of modelling are partially completed and solutions to the investigations are simplistic with minimal, inconsistent, incomplete or no explanations. Major errors distract the meanings.	Student demonstrates a general understanding of the fundamental mathematical concepts and applies them satisfactorily to similar situations. Most of the steps of modelling are adequately completed and solutions to the investigations (including part of the extensions) are presented with some explanations. Minor errors do not distract from the meaning.
Overall Mathematics Performance	*Does not achieve any requirement of the task.* Student work demonstrates minimal mathematical thinking and little or no understanding of mathematical ideas. Responses show little or no accomplishment of mathematical tasks.	*Requirements of the task not completed.* Student work demonstrates limited mathematical thinking and little understanding of mathematical ideas. While response to some basic elements are sometimes correct, student work often falls short of providing workable solutions.	*Limited Completion of the requirements of the task.* Student work demonstrates some mathematical thinking and partial understanding of mathematical ideas. Some responses are correct; however, gaps are evident and representations (for example, tables, formulae, equations, graphs) need elaboration.

SCORING SCALES FOR THE FOUR OUTCOME CATEGORIES FOR MAT4A TEACHING UNIT

Outcome Categories	*Level 4*	*Level 5*	*Level 6*
Communication			
Technology			
Application and Comprehension	Student demonstrates good understanding of all of the mathematical concepts of modelling and applies them successfully to new situations. All of the steps of modelling are consistently completed and solutions to most parts the investigations (including extensions) are presented with clear, coherent and unambiguous explanations.		
Overall Mathematics Performance	*Substantial completion of the requirements of the task.* Student work demonstrates substantial mathematical thinking and understanding of essential mathematical ideas. Responses meet most expectations; they are basically complete and correct although the work may contain minor flaws; and, include appropriate representations (for example, tables, formulae, equations, graphs).	*Full achievement of the requirements of the task.* Student work demonstrates solid mathematical thinking and full understanding of mathematical ideas. Responses fully meet expectations; they are usually correct and complete; and use appropriate representations (for example, tables, formulae, equations, graphs) and technology.	*Solid work that may go beyond the requirements of the task.* Student work demonstrates rigorous mathematical thinking and in-depth understanding of essential mathematical ideas. Responses meet and often exceed expectations; they are consistently correct and complete, and use appropriate representations (for example, tables, formulae, equations, graphs) and technology.

SCORING SCALES FOR THE FOUR OUTCOME CATEGORIES FOR MTT4G TEACHING UNIT

Outcome Categories	*Level 1*	*Level 2*	*Level 3*
Communication	The performance demonstrates a fragmented organization of the project which prevents the communication of the investigation to the readers. Data, results and conclusions are presented with minimal or no support.	The communication demonstrates adequate organization of the project to communicate most of the elements of the investigation to the readers. Data, results and conclusions are presented with limited support.	The performance demonstrates an effective organization of the project to communicate the investigation clearly to the readers. Data, results and conclusions are clearly presented and supported.
Mathematical Conventions	The performance demonstrates minimal command of the technology. The report of the project are prepared using the minimal capabilities of the software.	The performance demonstrates adequate command of the technology. Parts of the report of the project are prepared using some fundamental capabilities of the software.	The performance demonstrates a good command of the technology. The report of the project are effectively prepared using the full capabilities of the software.
Application and Comprehension	Student demonstrates minimal grasp of the very basic mathematical concepts and little evidence of applying them to other situations. Most of the steps of modelling are unstructured, incomplete and solutions to investigations are not understandable or omitted.	Student demonstrates partial understanding of some basic mathematical concepts of modelling and often unable to apply them to new situations. Some basic steps of modelling are partially completed and solutions to the investigations are simplistic with minimal, inconsistent, incomplete or no explanations. Major errors distract the meanings.	Student demonstrates a general understanding of the fundamental mathematical concepts and applies them satisfactorily to similar situations. Most of the steps of modelling are adequately completed and solutions to the investigations (including part of the extensions) are presented with some explanations. Minor errors do not distract from the meaning.
Overall Mathematics Performance	*Does not achieve any requirement of the task.* Student work demonstrates minimal mathematical thinking and little or no understanding of mathematical ideas. Responses show little or no accomplishment of mathematical tasks.	*Requirements of the task not completed.* Student work demonstrates limited mathematical thinking and little understanding of mathematical ideas. While response to some basic elements are sometimes correct, student work often falls short of providing workable solutions.	*Limited Completion of the requirements of the task.* Student work demonstrates some mathematical thinking and partial understanding of mathematical ideas. Some responses are correct; however, gaps are evident and representations (for example, tables, formulae, equations, graphs) need elaboration.

SCORING SCALES FOR THE FOUR OUTCOME CATEGORIES FOR MTT4G TEACHING UNIT

Outcome Categories	*Level 4*	*Level 5*	*Level 6*
Communication			
Mathematical Conventions			
Application and Comprehension	Student demonstrates good understanding of the mathematical concepts involved in the investigations. All of the appropriate calculations are shown and conclusions and answers to questions are presented in clear and correct ways.		
Overall Mathematics Performance	***Substantial completion of the requirements of the task.*** Student work demonstrates basic understanding of essential mathematical ideas. Responses are usually complete and correct although the work may contain minor flaws; and, including appropriate representations (tables, formulae, equations, graphs).	***Full achievement of the requirements of the task.*** Student work demonstrates full understanding of essential mathematical ideas. Responses fully meet expectations; they are usually correct and complete; and include appropriate representations (tables, formulae, equations, graphs).	***Solid work that may go beyond the requirements of the task.*** Student work demonstrates a thorough understanding of essential mathematical ideas. Responses meet and often exceed expectations; they are consistently correct and complete, and have include appropriate representations (tables, formulae, equations, graphs).

For the paper-and-pencil tests, there are three levels of difficulty for each Grade 12 course. While there will be differences in the specific questions for each course with respect to content, language and structure, the descriptions of the three levels are common.

LEVEL 1 - 90% of all students should be able to complete the tasks at this level. Students will:

- complete paper-and-pencil manipulative skill work;
- perform routine one-step problems;
- solve problems categorized by type;
- answer questions that require, for example, yes/no, a word, a simple sketch, definition, or a number as a response.

LEVEL 2 - 70% of all students should be able to complete the tasks at this level. Students will:

- solve word problems that involve more than one step;
- use explicitly stated strategies to solve problems within the range of specified mathematical content;
- recognize equivalent representations of the same concept;
- prepare solutions in full and proper form, using the appropriate notation, organizational skills and mathematical structure.

LEVEL 3 - 30% of all students should be able to complete the tasks at this level. Students will:

- solve non-routine problems;
- apply mathematical processes, skills and techniques in novel problem situations;
- connect mathematics to other subjects and to the world outside the classroom;
- present solutions descriptively, numerically, graphically, geometrically, or symbolically selecting the most appropriate mode and using rigorous mathematical form, terminology and representations.

THE SCARBOROUGH CONTEXT

Every secondary Scarborough school offering Grade 12 mathematics courses participated in the Mathematics Review in March-June, 1996. Semestered secondary schools offering mathematics courses also participated in the Review in November-January, 1997. For the 1995/96 Mathematics Review, a total of 23 secondary schools with 80 Grade 12 MAT4A classes, 38 MTT4G classes and 4 MTL4B/MTW4B classes, participated. Ninety teachers teaching 122 Grade 12 mathematics courses in 1995/96 were asked to administer the review to their students. Seventy-one teachers (79% return rate) teaching 103 classes (84% return rate) completed the teacher questionnaires and opportunity-to-learn forms (O.T.L.). There were 2486 Grade 12 students tested in 1995/96, including all ESL/ESD, and Special Education students as identified by themselves or their teachers.

School Characteristics

The information on schools was collected from the school questionnaires completed by the principal or his/her designate. Slightly more than one-half of the principals of Collegiate Institutes (C.I.'s) (return rate of 53%) and all principals of Business and Technical Institutes (B.T.I.'s) and High Schools (H.S.'s) (return rate of 100%) returned their questionnaires for the 1995/96 Review. Because of the differences between C.I.'s and B.T.I.'s and H.S.'s, they were grouped and analysed separately.

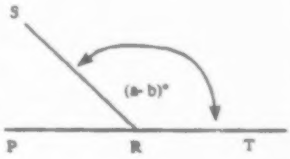
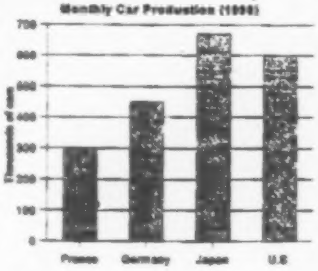
- About three-quarters of the respondents were C.I. principals (76%) and one-fifth were B.T.I. and H.S. principals (20%).

- Slightly more than one-half of the C.I.'s (54%) have almost all of their courses semestered while all of the B.T.I.'s and H.S.'s (100%) have almost all their courses semestered.

- The average total number of students in the C.I.'s (1393 students) was almost three times the average total number of students in B.T.I.'s and H.S.'s (495 students).

- On average, 12% of the total student population for C.I.'s was classified as ESL/ESD and 3% was classified as special education students. In the B.T.I.'s and H.S.'s, the average percentage of ESL/ESD students was 14% and of special education students was 25% of the total student population.

SAMPLE ITEMS
Paper-and-Pencil Instruments

MAT4A	*MTT4G/MTB4G*	*MTL4B/MTW4B*
***Level 1** - Algebra* *Determine if $x=2/3$ is a root of $3x^2-5x-2=0$.* *Show your work.*	***Level 1** - Geometry* *Circle the expression that correctly represents the measure of ∠ SRP.* *(A) $180° - a - b$* *(B) $a - b - 180°$* *(C) $180° - a + b$* *(D) $180° + a - b$* 	*General Numeracy* *Using the graph below, estimate the monthly car production in Germany in 1990.* 
***Level 2** - Trigonometry* *Using $\tan x = a/b$, find a simplified expression for $\tan 2x$*	***Level 2** - Trigonometry* *Δ ABC has sides a, b, c. Solve the triangle given $a = 12$ cm, $b=17$ cm and $\angle C = 120°$*	*You and Your Money* *A typical family spends 9% of gross income for transportation. If the family's income is $35 000, what is the amount spent for transportation?*
***Level 3** - Logarithms and Exponents* *Solve for x:* $\frac{\log_2 x}{\log_4 2x} = \frac{\log_8 4x}{\log_{10} 8x}$	***Level 3** - Force and Vectors* *The intended destination for a flight is 2400 km away from an airport. The airspeed of the airplane is 650 km/h, and its heading N60°E. A wind of 55 km/h from the southeast affects the plane's velocity. Find the arrival time (to the nearest minute) if take-off was at 7:35a.m.*	*Math on the Job* *A kitchen floor 3.8 m by 4.7 m will be covered with tiles. It takes 16 tiles to cover 1 square metre. Find the approximate number of tiles to cover the whole floor.* *A Place of Your Own* *Rent charges should not be more than 30% of one's salary. What should be the maximum rent for a person making $2 600 per month?*

- More special programs were offered by the B.T.I.'s and H.S.'s than C.I.'s except the French Immersion/Extended French program which was offered by 9% of the C.I.'s but not offered by the B.T.I.'s and H.S.'s (0%).
  - About two-fifths of the B.T.I.s and H.S.'s (40%) offered Adult programs while one-quarter of the C.I.'s (27%) offered Adult programs.
  - In addition, two-fifths of the B.T.I.s and H.S.'s (40%) offered Booster classes and three-fifths of the B.T.I.'s and H.S.'s (60%) offered special programs to their students compared with about one-tenth of the C.I.'s (9%).
- On average, three times more mathematics sections (Grade 9 to OAC) were offered by the Mathematics Department of C.I.'s (57) than B.T.I.'s and H.S.'s (18) in 1995-96.
- At the Grade 12 level, the average number of mathematics sections offered by the C.I.'s (22 MAT4A courses and 11 MTT4G courses) were ten times more than the sections offered by B.T.I.'s and H.S.'s (2 MTT4G courses and 1 MTL4G/MTW4B).
- There were no MAT4A courses offered by the B.T.I.'s and H.S.'s.
- The average number of full-time mathematics teachers in the C.I.'s (8) were more than those in the B.T.I.'s and H.S.'s (2) which depended more on part-time mathematics teachers (8).
- The average percent of annual school budget that was allocated to mathematics was 3% for C.I.'s and 1% for B.T.I.'s and H.S.'s.
- School principals estimated that the rate of absenteeism among MAT4A students to be 9% and MTT4G/MTB4G students to be 13-14% in all school types.
- Eighty percent or higher of all students (80% of C.I. students and 87% of B.T.I. and H.S. students) who initially registered in a grade 12 mathematics course would stay on to successfully complete that course.
- Principals of C.I.'s expected an average of 35% of student graduates would go to community colleges and 51% would go to universities. Principals of B.T.I.'s and H.S.'s expected 11% and 0% of their students to follow these respective routes.

Teacher Characteristics

Of all the Grade 12 mathematics teachers surveyed, 71 teachers (return rate of 79%) teaching 103 Grade 12 Mathematics sections returned their completed questionnaires for the 1995/96 Review.

- Most of the teachers who returned the teacher questionnaires were from C.I.'s (93%). 7% were from B.T.I.'s and H.S.'s.
- More male teachers taught mathematics in the B.T.I.'s and H.S's (75%) than C.I.'s (53%).
- Most of the teachers teaching Grade 12 mathematics were ones with 20+ years of teaching experience (68%). Most of their total teaching experience was gained in teaching mathematics. Only 5% of the Grade 12 mathematics teachers had between 0 to 5 years of teaching experience.

- In the past 10 years, two-thirds of the teachers teaching the MAT4A course had taught this course more than 5 times (66%). There was a smaller percentage of teachers teaching the MTT4G course (33%) and the MTB4G course (14%) more than 5 times. All of the teachers teaching MTW4B/MTL4B had taught these courses (100%) fewer than 5 times.

- Slightly over three-quarters of the teachers in C.I.'s indicated that they had specialized in mathematics in university (77%) while one-half of teachers of B.T.I.'s and H.S.'s stated that their specialization in university was mathematics (50%).

- Most of the C.I. teachers were qualified in senior division mathematics (83%) and 64% had their mathematics specialist. In addition, 23% had a post graduate masters degree, and 18% were qualified in computer studies.

- One-half of the B.T.I. and H.S. teachers were qualified in senior division mathematics (50%), and/or had their mathematics specialist (50%). In addition, 50% had post graduate masters degree and 50% were qualified in computer studies.

- About one-quarter of the teachers said that they were members of professional mathematics organizations (e.g., SAME 25%, OAME 24% or NCTM 11%). 64% of C.I. teachers and 75% of B.T.I. and H.S. teachers belonged to no professional mathematics organization.

- The three most valuable sources of professional development experiences that C.I. teachers had during the past five years were classroom experiences (24%), advice of colleague (22%), and Program Department Inservice workshops/seminars (15%). The most valuable sources of professional development experiences for B.T.I. and H.S. teachers were Program Department Inservice workshops/seminars (25%), school (17%), university (17%), and classroom experience (17%).

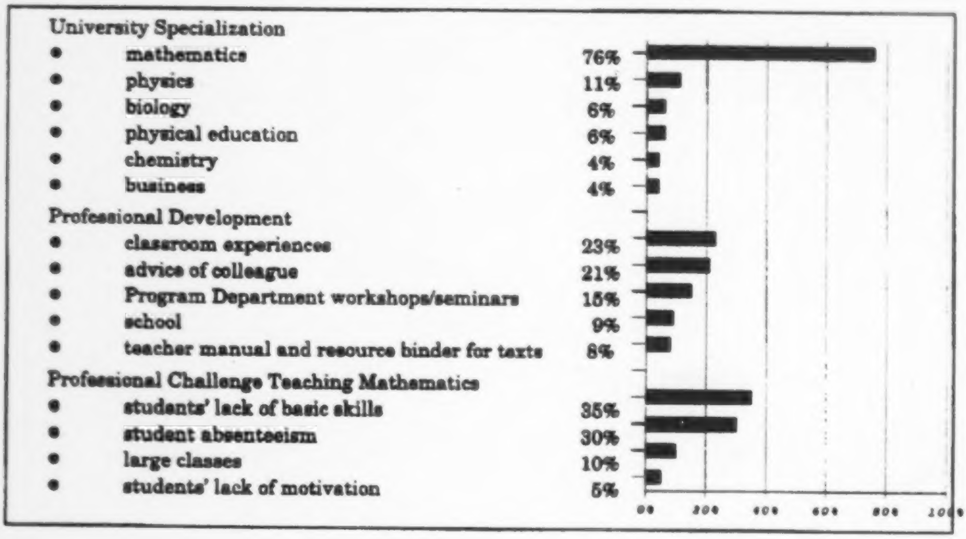


Fig. 1 Teacher Training and Challenge

- The greatest professional challenge that all teachers claimed to have in teaching grade 12 mathematics were students' lack of basic skills (35% for C.I. teachers, 38% for B.T.I. teachers), student absenteeism (30% for C.I. teachers, 25% B.T.I. teachers), large classes (10% for C.I. teachers, 13% B.T.I. teachers), and lack of computers (6% for C.I. teachers,13% B.T.I. teachers). In addition, Students' lack of motivation (10%) were other challenges cited by C.I. teachers while total teaching workload (13%) were the other challenges facing B.T.I. and H.S. teachers.

- About one-fifth of the C.I. teachers (21%) said that they had visited the Mathematics Centre during the school year and 15% of the teachers found the resources of value.

Classroom Characteristics

The information on classroom characteristics was collected from the 103 questionnaires returned by the teachers.

- 96% of the classrooms were in C.I.'s and 4% of the classrooms were in B.T.I.'s and H.S.'s.

- There were more MAT4A courses (70%) offered by the schools than MTT4G/MTB4G courses (30%).

- Slightly more mathematics courses were semestered (55%) than full year (43%) in C.I.'s. All mathematics courses in B.T.I.'s and H.S.'s were semestered (100%).

- The average number of students in a C.I. mathematics classroom was 21, and in a B.T.I. and H.S. mathematics classroom, it was 15.

- There were similar average numbers of male (12) and female students (11) in C.I. mathematics classrooms, and in B.T.I. and H.S. mathematics classrooms (8 male and 7 female students).

Student Characteristics

A total of 2284 Grade 12 students completed the student questionnaires in 1995/96. When interpreting the information, readers are cautioned that 98% of the respondents were from C.I.'s (return rate of 93%) and 2% from B.T.I.'s and H.S.'s (return rate of 58%). In fact, none of the students from the two MTL4B/MTW4B courses returned the student questionnires.

- Three-quarters of the students from the C.I.'s were taking MAT4A courses (76%) and the remaining (24%) were taking MTT4G courses. All students from B.T.I.'s and H.S.'s who completed the questionnaires were taking MTT4G courses.

- 5% of students from C.I.'s (5%) were re-taking a Grade 12 course while 13% of students from B.T.I.'s and H.S.'s were re-taking a Grade 12 course.

- Approximately one-third of the students obtained a grade of 80% or higher in the last math course (31%) they had taken. Two-thirds obtained a grade between 50 and 79% in the last math course (66%).

- 87% of students completed their last math course at a regular Scarborough day school, 6% reported earning the credit at a Scarborough summer school, and only 1% reported earning the credit at night school.

- 10% of the students had a gap of more than one semester between their last math course and their current Grade 12 math course.
- There were slightly more male students (52%) than female students (48%).
- 72% of the students from C.I.'s were born on or after 1978 compared to only 12% of the B.T.I. and H.S. students who were born on or after 1978.
- The average number of years that students had attended secondary schools was 3.8 years.
- There were more students who said they spoke other languages most often at home taking grade 12 math courses from the B.T.I.'s and H.S.'s (56%) than students from C.I.'s (35%).
- On average, 11% of the C.I. students taking grade 12 mathematics were classified as ESL/ESD and 3% of the students were classified as special education students. In the B.T.I.'s and H.S.'s, the average percentage of mathematics students who were classified as being ESL/ESD was 27% and for those who were special education students, the figure was 3%.
- A number of the students in the B.T.I.'s and H.S.'s were attending adult classes (19%) compared to none in the C.I.'s (0%).
- 69% of the students watched more than 1 hour of TV programs each day at home.
- About one-half of the students had a part-time job (46%), with 21% working between 1 to10 hours per week, 18% working 11 to 20 hours and 7% working more than 20 hours.
- Most C.I. students (91%) planned to attend a community college or a university while 1% of students from B.T.I.'s and H.S.'s had the same plan.

CLASSROOM INSTRUCTION AND PRACTICE

Instructions

- More than half of the instructional periods for the whole course (average 83 periods) were spent on Lecturing/Direct Instruction (57%), while other periods were spent on review for tests or exams (11%), tests (10%), student projects and/or investigations (8%), quizzes (7%) and other (2%)
- Teachers estimated that during a typical class (average 74 minutes), more time (in minutes) was spent on applying the Socratic method (18), followed by helping individual students (17), and taking up homework (15). Other time was spent on lecturing (9), supervising group work (9), taking attendance and other administrative duties (6) and other (2).
- Student's term mark was determined mostly by final/term exam (33%), unit tests (33%), and quizzes (11%).
- For 46% of the courses, final exams were optional. The grade students had to maintain to get exemption ranged from 50% to 70%.

- Students from about one-half of the classrooms were assigned less than 1 hour of homework each day (53%), 32% were assigned 1-2 hours of homework, while very few were assigned more than 2 hours of homework every day.

- The textbooks used by MAT4A classes were Mathematics 12 (30%), Applied Mathematics 12 (2%) and others (47%). The textbooks used by MTT4G classes were Math Matters Book 4 (75%), Applied Mathematics 12 (6%) and others (13%).

Use of Technology

- Over 80% of teachers said that their students used a scientific calculator (83%) in their math class almost everyday. About one-half said that their students used a computer (52%) once in a while. About one-third of the teachers said that their students never used a computer (30%) and over one-half said their students never used a graphing calculator (57%) in class.

- On average, there were 29 PC's and 15 Macintosh's available for students' use in each school.

- The computer programs that teachers found most useful were Zap-A-Graph (53%) and ClarisWorks/Microsoft/PC Spreadsheets (44%).

- Other resources and/or visual aids that teachers used in their grade 12 mathematics classes were an overhead projector (57%), alternate text books (48%), newspapers/ magazines (35%), games (21%) and videos (20%).

STUDENT ATTITUDES TOWARDS MATHEMATICS

Attitudes towards Mathematics

- Most students strongly agreed or agreed that they liked mathematics (70%).

- Students recognized that mathematics helped them with other courses (70%), helped them at work (76%), and was an important skill in daily life (86%).

- About one-half of the students thought they were good at mathematics (56%) or mathematics was not difficult for them (48%).

- A large majority of students strongly agreed or agreed that there was usually more than one way to solve a mathematical problem (91%), but the main trouble some students had in solving mathematical problems was understanding what the sentences said (40%).

- The main strategies that students used when they did not understand a problem in mathematics included trying different ways on their own (61%), asking another student for help (67%), and asking their teacher for help (54%).

Mathematics Learning

- In the mathematics class, almost all students said that they listen to the teacher presenting to the whole class (89%), used a scientific calculator (87%), and work at solving mathematics problems on their own (69%) almost everyday.

- In addition, a lot of students said that they wrote quizzes (46%), wrote term tests or unit (topic) tests (23%), and worked at solving mathematics problems in small groups (20%) once a week in the mathematics class.

- Most students indicated that they used a scientific calculator to help with their mathematics homework almost everyday (81%). Very few students said that they used a graphing calculator (3% almost everyday, 2% once a week), a computer (2% almost everyday, 3% once a week), or got help from someone in their family (7% almost everyday, 10% once a week).

- On average, two-fifths of the students spent less than 1 hour on mathematics home work each day (43%) than between 1-2 hours (39%). Very few students spent more than 2 hours (11%) or never spent anytime on homework (6%).

- About two-fifths of the students spent between 1-2 hours to study for a term test in mathematics (38%), than less than 1 hour (19%), 2-3 hours (24%) or more than 3 hours (14%).

- Above one-half of the students said that they spent more than 3 hours studying for an exam in mathematics (52%), with fewer students spending between 2-3 hours (22%), between 1-2 hours (17%) or less than 1 hour (7%).

Fig. 2 Mathematics Learning

In Your Math Class, how often do you:	Almost Everyday	Once a Week
• listen to your teaching presenting to the whole class	89%	4%
• use a scientific calculator	87%	
• work at solving mathematics problems on your own	69%	15%
• write quizzes	12%	46%
• work at solving mathematics problems in small group	15%	20%
• write term tests or unit (topic) tests	5%	23%
• do mathematics in front of the whole class?	12%	13%
• use a computer	9%	7%
• work on mathematics projects	6%	9%
• use a graphing calculator	3%	2%

LEARNING MATHEMATICS IN SCHOOL AND ATTITUDES TOWARDS THE COURSE

- When students were asked about how they would like to learn mathematics in school, most preferred to listen to the teacher (81%), followed by to work on mathematics problems from textbooks (60%), to solve mathematics problems in small groups (48%), and to read from textbooks (40%).

- About two-fifths of the students from C.I.'s felt that the mathematics course this year was just right for them (40%), slightly more students found the course difficult or very difficult (42%), and fewer students found it easy or very easy (17%).

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- About two-fifths of the students from B.T.I.'s and H.S.'s who found the mathematics course this year easy or very easy (38%), about one-third of the students found it just right for them (34%), and fewer students found the course difficult or very difficult (27%).

STUDENT ACHIEVEMENT

The achievement of students in mathematics were evaluated by grading demonstrated performance on a two-week assessment unit and paper-and-pencil test items that covered the core content areas articulated in the *OS:IS Guidelines for Mathematics (1985)*.

Achievement at the MAT4A Level

Teaching Unit

The focus of the teaching unit was on mathematical modelling to elicit appropriate information and to make relevant predictions for future events. The completion rate of the exercises and quizzes included in the teaching unit was between 83% and 98%.

Student achievement in the teaching unit was measured against holistic scoring scales developed by Scarborough educators for four outcome categories *communication, technology, mathematics applications* and *overall mathematics performance* (See figure 3).

All students are expected to achieve at level 2 and higher in the categories *communication* and *technology*. Students are expected to achieve level 3 and higher in *mathematics applications*. In the category *overall mathematics performance*, students are expected to achieve level 4 and higher.

Fig. 3 **Summary Student Achievement for MAT4A Teaching Unit**

Outcome Categories	*Expected Level*	*Higher Level*
Communication	*Level 2+* *74%*	*Level 3* *21%*
Technology	*Level 2+* *83%*	*Level 3* *11%*
Mathematics Applications	*Level 3+* *39%*	*Level 4* *14%*
Overall Mathematics Performance	*Level 4+* *33%*	*Level 5+* *7%*

- Approximately three-quarters of the students achieved level 2 and higher (74%) and one-fifth of the students achieved level 3 (21%) in *communication*.
- Most students performed at level 2 and higher (83%) in the technology category but only 11% of the students performed at level 3.
- Close to two-fifths of the students achieved level 3 and higher (39%) and 14% attained level 4 in the category mathematics applications.

- In the *overall mathematics performance* category, 33% of the students achieved level 4 and higher while 7% achieved level 5 and higher.

Paper-and-Pencil Test

A variety of question types were included which required students to solve problems and provide full solutions in the five core content areas: *algebra, logarithms and exponents, the algebra of functions, trigonometry* and *cumulative numeracy*.

Achievement

- Over one-half of the students correctly answered the level 1 items (56%) on *Algebra* and about one-third of the students (30%) correctly completed the level 2 items. One-seventh of the students were able to provide partially correct answers to both level 1 (14%) and 2 items (15%).

- For the items on *Logarithms and Exponents*, more than three-fifths of the students were able to answer the level 1 questions (62%) and 6% more gave partially correct answers. Slightly more than one-quarter of the students were able to answer the level 2 questions (28%) and another 10% were able to give partially correct answers.

- Close to one-half of the students accurately answered the level 1 items (48%) on *Algebra of Functions* and another 10% of the students were able to complete the questions partially. About one-third provided correct answers to the level 2 items (30%) while an additional 12% provided partial solutions.

- Slightly less than one-half of the students correctly answered the level 1 items (47%) on *Trigonometric Functions* and another 16% gave partial answers to the questions. More than one-quarter of the students (27%) correctly answered the level 2 items and another 7% of the students gave partial solutions to the items.

- *Cumulative Numeracy* items included basic numeracy skills accumulated through the lower grades. Fewer than one-half of the students correctly answered the level 1 items (44%) and 3% more provided partial solutions. About one-third of the students answered the level 2 items (31%) and another 9% of the students answered these items partially.

- Only 10% of the students were able to complete the level 3 items of all strands accurately and another 5% provided partial solutions to the questions.

Opportunity-to-Learn the Paper-and-Pencil Items

- Fewer than one-half of the teachers said that they had taught or reviewed the content of the *Algebra* items in the year of the review (34% for level 1 items and 48% for level 2 items). Close to one-half of the items were considered prerequisites for grade 12.

- The content area of *Logarithms and Exponents* was well addressed by most of the teachers in the year of the review (88% for level 1 items and 77% for level 2 items). About 10% of the items were assumed prerequisites for grade 12 by the teachers.

- In the content area *Algebra of Functions*, more than one-half of the teachers said that they had taught or reviewed the level 1 items (55%) and slightly more teachers said that they had taught or reviewed the level 2 items (68%). 26% of the level 1 items and 11% of the level 2 items were considered prerequisites for grade 12 by the teachers.

- The content area of *Trigonometry* was addressed by more than 80% of the teachers (82% for level 1 and 81% for level 2 items). Less than 10% of the items were prerequisites for grade 12.
- Very few teachers reviewed the items in *Cumulative Numeracy* (3% level 1, 18% level 2). 80% and more of the items were assumed prerequisites for grade 12.
- Slightly more than one-third of the level 3 items (35%) of all of the strands were addressed by the teachers and 21% of the items were considered prerequisites for grade 12.

Fig. 4 Summary Student Achievement MAT4A Paper-and-pencil Test

Category (Expected performance)	Correctly or Partially Answered	Correctly Answered	Opportunity-to-Learn	
			(Taught or reviewed)	(Assumed Prerequisites)
Algebra				
• Level 1 (90%)	70%	56%	34%	64%
• Level 2 (70%)	45%	30%	48%	43%
Logarithms and Exponents				
• Level 1 (90%)	68%	62%	88%	11%
• Level 2 (70%)	38%	28%	77%	12%
Algebra of Functions				
• Level 1 (90%)	58%	48%	55%	26%
• Level 2 (70%)	42%	30%	68%	11%
Trigonometric Functions				
• Level 1 (90%)	63%	47%	82%	9%
• Level 2 (70%)	34%	27%	81%	6%
Cumulative Numeracy				
• Level 1 (90%)	47%	44%	3%	87%
• Level 2 (70%)	40%	31%	18%	79%
All Strands				
• Level 3 (30%)	15%	10%	35%	21%

ACHIEVEMENT AT THE MTT4G/MTB4G LEVEL

Teaching Unit

The focus of the general level teaching unit was the Stock Market. This unit actively involved students in the tracking of stock prices from newspapers and then creating charts and graphs to track the stocks.

All students are expected to achieve at level 2 and higher in the categories *communication* and *mathematics conventions*. Students are expected to perform at level 3 and higher for *mathematics applications*. In the category *overall mathematics performance*, students are expected to achieve level 4 and higher.

- Two-thirds of the students achieved level 2 and higher (65%) and one-sixth of the students achieved level 3 (16%) in *communication*.

- Over 80% of the students performed at level 2 and higher (83%) in the *mathematics conventions* category and 34% of the students performed at level 3.

- Slightly less than two-fifths of the students achieved level 3 and higher (39%) in the category *mathematics applications* ehile very few students attained level 4 (9%).

- In *overall mathematics performance*, 25% of the students achieved level 4 and higher while 2% achieved level 5 and higher.

Fig. 5 Summary Student Achievement for MTT4G/MTB4G Teaching Unit

Outcome Categories	***Expected Level***	***Higher Level***
Communication	*Level 2+* *65%*	*Level 3* *16%*
Mathematics Conventions	*Level 2+* *83%*	*Level 3* *34%*
Mathematics Applications	*Level 3+* *39%*	*Level 4* *9%*
Overall Mathematics Performance	*Level 4+* *25%*	*Level 5+* *3%*

Paper-and-Pencil Test

A variety of question types were included that required students to solve problems and provide full solutions in the four core content areas: *Numerical and Algebraic Methods*, *Geometry*, *Trigonometry* and *Forces and Vectors*.

Achievement

Overall, less than 40% of the students gave correct or partially correct answers to the questions in all four content areas tested.

- The highest achievement was in *Numerical and Algebraic Methods* where more than one-third of the students were able to give correct (30%) or partially correct (7%) solutions to the level 1 items. Slightly lower percentage of students gave correct (23%) or partially correct (9%) solutions to the level 2 items.

- Slightly more than one-quarter of the students answered the level 1 items in *Geometry* correctly (22%) or partially correctly (5%). Close to one-quarter of the students answered the level 2 items correctly (14%) or partially correctly (9%).

- The percentage of students who could answer the level 1 *Trigonometry* items was 29% (24% correctly, 5% partially correctly) and 23% for the level 2 items (15% correctly, 8% partially correctly).

- About one-quarter of the students answered the level 1 items in *Forces and Vectors* correctly (19%) or partially correctly (5%) while less than one-sixth answered the level 2 items correctly (7%) and partially correctly (7%).

- Very few students could answer the level 3 items of all strands (3% correctly, 4% partially correctly).

Opportunity-to-Learn the Paper-and-Pencil Items

- Approximately one-half of the teachers claimed that they had taught or reviewed the items in Numerical and Algebraic Methods (42% for level 1 and 52% for level 2). Many teachers assumed these items were prerequisites for the course (56% level 1 and 32% level 2).

- Similar percentage of teachers said that they had taught the items in Geometry (44% for level 1 and 43% for level 2). About one-third of the items were considered prerequisites (35% level 1 and 29% level 2).

- The content area of Trigonometry was covered by most of the teachers. About 80% of the teachers said that they had taught or reviewed the level 1 (83%) and level 2 items (76%).

- Slightly more than one-half of the teachers said that they had taught or reviewed the items in Forces and Vectors (54% for level 1 and 55% for level 2).

- Slightly more than one-half of the teachers said that they had taught or reviewed the level 3 items in all of the strands (53%) in the year of the review.

Fig. 6 Summary Student Achievement MTT4G/MTB4G Paper-and-pencil Test

Category (Expected Performance)		Correctly or Partially Answered	Correctly Answered	Opportunity-to-Learn (Taught or reviewed)	(Assumed Prerequisite)
Numerical and Algebraic Methods					
• Level 1	(90%)	37%	30%	42%	56%
• Level 2	(70%)	32%	23%	52%	32%
Geometry					
• Level 1	(90%)	27%	22%	44%	35%
• Level 2	(70%)	23%	14%	43%	29%
Trigonometry					
• Level 1	(90%)	29%	24%	83%	6%
• Level 2	(70%)	23%	15%	76%	4%
Forces and Vectors					
• Level 1	(90%)	24%	19%	54%	8%
• Level 2	(70%)	14%	7%	55%	7%
All Strands					
• Level 3	(30%)	7%	3%	53%	8%

ACHIEVEMENT AT THE MTL4B/MTW4B LEVEL

Teaching Unit

The focus of the basic level teaching unit was on *Percentages* and *Interest Rates*. Students were given step-by-step instructions to solve problems related to the handling of money using everyday life examples. Included in the two-week teaching units were exercises, quizzes and two unit tests.

- The items included in Unit Test #1 were simple problems involving percentages, decimals and fractions; 52% of the students correctly answered the items and 15% gave partial solutions to the items.
- There were more word problems and more complex calculations involving fractions, decimals or percents in Unit Test #2; 26% of the students provided correct answers and 20% provided partial answers to the questions.

Paper-and-pencil Test

Achievement

In the paper-and-pencil test, student performance in the content areas tested was 33% in *You and Your Money*, 30% in *General Numeracy* , 28% in *A Place of Your Own* and 25% in *Math on the Job*. No students were able to complete any of the level 3 items (0%).

Opportunity-to-Learn the Paper-and-Pencil Items

- Teachers said that they taught or reviewed 31% of the items on *General Numeracy*. 20% of the items were considered prerequisites for grade 12.
- Most of the items in You and Your Money (69%), Math on the Job (71%), and A Place of Your Own (74%) were taught or reviewed by the teachers. Some of the items were assumed pre-requisities for the course.
- All of the level 3 items for all strands were not taught or reviewed.

Fig. 7 Summary Student Achievement MTL4B/MTW4B Paper-and-Pencil Test

Category (Expected Performance)	**Correctly Answered**	**Opportunity-to-Learn**	
		(Taught or reviewed)	**(Assumed Prerequisite)**
General Numeracy • Level 1 (90%)	30%	31%	20%
You and Your Money • Level 1 (90%)	33%	69%	23%
Math on the Job • Level 1 (90%)	25%	71%	10%
A Place of Your Own • Level 1 (90%)	28%	74%	5%
All Strands • Level 3 (30%)	0%	0%	0%

CONCLUSIONS

The Grade 12 Review of Mathematics has provided a rich body of empirical data that will enable Scarborough's mathematics educators to make informed and reflective decisions about future directions with respect to the continuation and implementation of the sound pedagogical requirements and process components specified in both the *OS:IS Guidelines for Mathematics (1985)* and the *NCTM Curriculum and Evaluation Standards for School Mathematics (1995)*.

When interpreting the findings, it must be recognized that this was the first time that mathematics teachers from the secondary panel has been involved in a performance-based, holistically-scored review of mathematics. The process, the format and the outcome categories assessed in the Teaching Unit were new for everyone involved, from the Research and Program staff, to teachers and students.

In assessing the Advanced-level (MAT4A) and the General-level (MTT4G/MTB4G) *Teaching Units*, the Mathematics outcomes were grouped into the following four reporting categories:

Reporting Categories for Teaching Unit	Expected Performance Level
Communicating	Level 2+
Technology/Mathematics Conventions	Level 2+
Mathematics Applications	Level 3+
Overall Mathematics Performance	Level 4+

Student achievement in the MAT4A Teaching unit was highest in technology (83% at level 2+), followed by communication (74% at level 2+), mathematics applications (39% at level 3+), and overall mathematics performance (33% at level 4+).

In the MTT4G/MTB4G Teaching unit, students performed well in mathematics conventions (83% at level 2+), followed by communication (65% at level 2+), mathematics applications (39% at level 3+), and overall mathematics performance (25% at level 4+).

In all three of the paper-and-pencil tests, the core content areas from the *OS:IS Guidelines* and the mathematics course descriptions from most Scarborough schools were included. It was expected that almost all (90%) of the students would correctly answer all level 1 questions; 70% would answer the level 2 questions and 30% would solve the level 3 items.

In the MAT4A Paper-and-pencil test, about 70% of the level 1 items in Algebra (70%) and Logarithms and Exponents (68%) were correctly or partially correctly answered by students. About 60% of the Level 1 items in Algebra of Functions (58%) and Trigonometric Functions (63%) were answered correctly or partially correctly. About one-half of the students provided partial or correct answers to the Level 1 items in Cumulative Numeracy (47%). Level 2 items in all of the strands were answered correctly or partially correctly by more than one-third of the students. 15% of the students provided correct or partial answers to the level 3 items of all strands.

The information collected from the Teacher Opportunity-to-Learn forms indicated that there were gaps in the coverage of the paper and pencil test items. MAT4A teachers addressed most of the paper-and-pencil items in Logarithms and Exponents (88% for level 1, 77% for level 2) and Trigonometry (82% for level 1, 81% for level 2). The coverage of items was lower in Algebra of Functions (55% for level 1, 68% for level 2) and Algebra (34% for level 1, 48% for level 2). Very few teachers reviewed the items in Cumulative Numeracy (3% for level 1, 18% for level 2). Level 3 items of all strands were taught or reviewed by one-third of the teachers (35%). About 80% of the items in Cumulative Numeracy and 50% of the items in Algebra were considered prerequisites for Grade 12 by the teachers.

Student achievement in the General Level MTT4G/MTB4G Paper-and-pencil test items was much lower. One-third of the students answered the level 1 items in Numerical and Algebraic Methods (37%) and Trigonometry (29%) correctly or partially correctly. About one-quarter of the students were able to complete the level 1 items in Geometry (27%) and Forces and Vectors (24%) correctly or partially correctly. 14% to 32% of the students could complete the level 2 items in all strands correctly and partially correctly. 7% of the students were able to complete the level 3 items in all strands correctly or partially correctly.

The information collected from the Teacher Opportunity-to-Learn forms indicated that there were large gaps in the coverage of the paper and pencil test items at the General level. MTT4G/MTB4G teachers reported that they reviewed most of the test items in Trigonometry (83% for level 1 items and 76% for level 2). However, the other content areas were taught or reviewed by about one-half of the teachers (42-55%). About 40% of the items in Numerical and Algebraic Methods and 30% of the items in Geometry were considered prerequisites for the course by the teachers.

In the Basic Level MTL4B/MTW4B Paper-and-pencil test, about one-third or fewer of the level 1 items were correctly answered by students (25-33%). None of the students were able to complete the level 3 items (0%) of all strands. Teachers teaching the MTL4B/MTW4B courses indicated on the opportunity-to-learn forms that they addressed or reviewed about 70% of the level 1 items in all content areas except General Numeracy (31%). They reported that none of the level 3 items were addressed in the basic level course. About 20% of the items in General Numeracy and You and Your Money were considered prerequisites for the course by the teachers.

Responses from the Teacher Questionnaire indicated that 76% of the teachers had mathematics specialization in university, 83% of the C.I. teachers were qualified in senior division mathematics with 64% holding specialist qualifications. 63% of the teachers did not belong to a professional mathematics organization. The valuable sources of professional development experiences cited by teachers were "classroom experiences" (23%), "advice of a colleague" (21%) and "Program Department work-shops/seminars" (15%). Findings from questionnaires on classroom instruction and practice showed that for 46% of the classes, final exams were optional provided that students maintained average grades ranging from 50% to 70%. In addition, students from over one-half of the classrooms were assigned less than 1 hour of homework in mathematics each day (53%). 43% of the students reported that they did less than one hour of home work per day.

Findings from the Student Questionnaire showed that 31% of the students earned grades above 80% in their last mathematics course and 66% received grades between 50 and 79%. Despite this, 35% of the grade 12 teachers cited "students' lack of basic skills" as a professional challenge in teaching mathematics. About three-quarters of the students indicated that they liked mathematics (70%) and higher percentages of students recognized that "mathematics is an important skill in daily life" (86%), "mathematics will help me at work" (76%) and "mathematics help with other courses at school" (70%). 40% of the students found that the main trouble they had in solving mathematical problems was "understanding what the sentences said". There were over one-third of the C.I. students (35%) and one-half of B.T.I. and H.S. students (56%) who said that they "spoke other languages at home". 11% of C.I. students and 27% of B.T.I. and H.S. students were ESL/ESD students.

The findings of the Mathematics Review clearly suggest that student achievement was higher when they were given opportunities to work on applications to solve real life problems in non-traditional instructional settings (i.e., in the teaching unit) as compared to the paper-and-pencil tests. Most of the students possessed the technical/computer skills and analytical skills necessary to tackle the challenging problems/activities in the teaching units. The findings reinforce the direction that, "the learning of mathematics should be guided by the search to answer questions - first, at an intuition, empirical level; then by generalization; and finally by justification (proof)" as stated in the NCTM Curriculum and Evaluation Standards.

The findings reaffirm that Mathematics in Secondary schools is in need of realignment with the OS:IS Guidelines and the NCTM Standards. Furthermore, the Review findings and the recommendations they elicit, reinforce the need to apply and extend the style of mathematics education currently being required by the Common Curriculum: Mathematics Standards K-9 to the specialization years program. The issues related to curriculum, classroom instruction and practice as well as the relationships between summative assessment and retention of skills and knowledge will be addressed in the Action Plan. The Action Plan will also identify areas where we should place emphasis in the planning of curriculum, allocation of resources and professional development. The availability of such rich baseline data about the elements directly affecting mathematics teaching and learning will enable the Mathematics Division of the Program Department to work with schools, Mathematics Heads and teachers in a creative and collaborative manner in order to improve mathematics education in Scarborough.

REFERENCES

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: The Council.

Ad Hoc Session 10

TEACHING OF GRAPH THEORY FOR HIGH SCHOOL AND COLLEGE

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I BACKGROUND

Graph Theory—Development and Characteristics

Graph Theory began in the year 1736, with an article by L. Euler that discussed "The Koenigsberg bridge problem" (Fujii [5], p. 1). After about a one hundred year break, the theory was revived due to problems in the field of science, like *electricity networks* (Kirchoff, 1847 [6], p. 3), *chemical isomers* (Cayley, 1857 [6], p. 5), the *four colors problem* that was presented by De-Morgan in 1850, as well as additional problems that were translated and presented in terms of graph theory (Harary [6], p. 2-4).

In the last half-century, development of the theory has accelerated in various fields, mathematical and non-mathematical, in research areas, and practical areas. For example, Operations Research, Linear Programming, Computer Science, Networks, Transportation, Communications, Economics, and much more (for detailed examples see section A.5, Chapter A, in Turgman (1996) [13]).

Graph theory is considered to be an elegant branch of mathematics, which is easy to learn, based on few principles, not requiring previous mathematical knowledge, and having many practical applications (see for example H.N.V. Temperly's book [12]). These characteristics of graph theory, the recommendations of The Committee on Undergraduate Math Programs (CUPM) [4] and many others, convinced us to develop a new systematic course on graph theory for high school.

But in fact there is no course like this, and all the existent material deals on puzzles and games like the bridge problem, the problems of the knight piece on the chess board and drawing using a single line (see the chapter "What Is Mathematics?" in Tamur [10], and Fujii, [5]), the booklet "Graphs, Polytops and Maps" [1] by A. Altshuler, which, according to him, is made for 10-12 meetings of youth activity groups. Even the book "Graphs and Their Uses" by O. Ore [8] is designed for excelling high school students. No written work, including the above, was accompanied by systematic research, which included experimentation and assessment in the field, at different learning levels. (In Israel there are 3 levels called - 3 units, 4 units and 5 units). Among the reasons for this are:

a. Students spend very little time on systematic proofs of geometry or induction.

b. Important theorems on graph theory which are in university textbooks rely on ability to prove, that the high school student doesn't have.

c. Theorems in university textbooks rely on knowledge and tools that the high school student doesn't have.

II GENERAL VIEW OF THE RESEARCH

Some Goals of the Research

We have constructed a course in graph theory for the high school level, which is "an integration of new subject and new teaching method"—one of the characteristics of researches of science teaching by P. Tamir ([11] p. 250). Its aims are:

a. Development and writing a course in graph theory, with an emphasis on the practical aspect—graph theory as a mathematical model for high school students at the 3, 4 and 5 levels.

b. Testing the program by Formative and Summative evaluation in the field (M. Scriven in [7]) so that the following questions will be answered: Can graph theory be taught? To what level of depth and of understanding is it possible or desirable to reach? Is it possible to do so within a reasonable time frame?

c. Based on the results, to suggest a syllabus and teaching styles for an elective course, with objectives, subjects and teaching methods.

d. To be aware of typical mistakes in learning the theory and difficulties in teaching it.

e. To check if learning this theory has an influence on the student's achievements in problem solving and his attitude towards learning mathematics.

f. To check learning achievements.

The Population of the Research

During the research we used several kinds of populations for any stage as follows:

1. Preliminary Experiments - two years

The Experience	N	The Population	Length of Experiment
Advanced study for teachers, Tel-Aviv University	40	Teachers of Junior High School	two days (8 hours)
High School in Ofakim	18	Pupils of 11th grade	40 hours (2 hours/week)
Youth Science Center 1st year Uni. of Jerusalem	25	Pupils of 7-9th grade	academic year (2 hours/week)
Youth Science Center 2nd year, Uni. of Jerusalem	17	Pupils of 9-11th grade	academic year (2 hours/week)
Seminar at Tel- Aviv University	25	Academic staff in science teaching	one day
total	125		

2. Main Experiments - two years

	Institution of Learning	Mark Group	Kind of Class	Class Level	N
First year	Kfar Maymon High School, 6 hours/week	A	11th grade	4-5 points	14
First year	Sapir College, 6 hours/week	B	Pre-Academic-Humanities	3 points	32
Second year	Kfar Maymon 6 hours/week	C	11th grade	4-5 points	20
Second year	Sapir College 2 hours/week	D	Pre-Academic-Humanities	3 points	25
Second year	Sapir College 2 hours/week	E	Pre-Academic-Natural Science	4 points	16
Second year	Sapir College 2 hours/week	F	Pre-Academic - Exact Science	5 points	16
total		6 groups		123 pupils	

3. Control Classes (High School, All levels, Two years)

1. Sapir College:	2 classes (each year)
2. Kfar-Maymon:	1 class (each year)
3. Ashdod City:	2 classes (each year)
The total number :	193 pupils.

4. Other groups - (after the main experiment)

Institution of Learning	N	The Population	Length of Experiment	The teacher
Weizman Institute of science - Rehovot	15	Youth Activity Group	One academic year (2 hours/week)	Eran
Tel-Aviv University	20	Youth Activity Group	One academic year (2 hours/week)	Dr. Shemer
Levinsky College, Tel-Aviv	35	Students College for Elem. School	One academic year (2 hours/week)	A. Turgman
Levinsky College, Tel-Aviv	10	Students College for Junior High S.	One academic year (2 hours/week)	A. Turgman
High School of Technology, Jerusalem	50	Student of Engineering	One academic year (2 hours/week)	Jonathan Stoop

Some Main Results of the Research

We believe that:

1. This research showed how it is possible to lower the level of an important, interesting and practical subject, to the level of high school.
2. Learning this subject improves students' attitudes towards the disciplinary sciences.
3. Learning this subject strengthens a student's ability to build models and realistic practical situations.

The Course and its Teaching Aims

Because graph theory has a very wide scope that certainly cannot be covered in the framework of a high school course, and other considerations— mathematical, didactic, the nature of the topics, and more, we chose subjects that covered various aspects of the theory and principles of its foundation. Subjects that have a variety of applications, such as puzzles and games. Subjects that can be explained to and practiced by high school students. These are the topics that were decided upon by the end of the project (see [15]):

a. Definitions, Examples and Basic Concepts - First meeting for the student with Graph Theory.

b. Paths and Circuits on Graph - Systematic discussion on the subjects: paths, connectivity and circuits (including Eulerian graphs and Hamiltonian graphs).

c. Trees - Definitions and basic properties, including *spanning* tree and *optimal* tree.

d. Networks - Discussion on directed graphs, shortest path, the One-way Street Problem, flows in networks and the Min-max Theorem.

e. Puzzles, Games and Graph Theory - Description of 7 puzzles and games, translating each of them to a graphic model, analysis each one and finding all the mathematical solutions by graph theory.

The Teaching Aims of the Course

Students who will learn the course will:

1. Know basic concepts and theorems of the theory in various topics.
2. Can solve non complicated problems connected to Graph Theory.
3. Know what is a mathematical-graphic model, its roles and applications.
4. Can translate practical problems to graphic model.
5. Can apply graph theory in problem solving.
6. Be familiar with mathematical discipline which has practical applications in various subjects.
7. Have enjoyable learning.

III DESCRIPTION OF THE THESIS AND SOME RESULTS

The flow chart (next page) describes the research step- by- step from the preliminary experiments until the final conclusions. Below is a brief description of the work (For details, see Turgman, Chapters A and B [13]).

A. Introduction - General background of graph theory and its importance to high school.

B. The Methodology - Characteristic of researches in science teaching and description of the stages of the work; preliminary experiments, the syllabus of the course, the main experiments, the textbook, the summaries of the research, the results and conclusions.

C. The Preliminary Experiments - Full report of each experiment (see [14] appendix 1.0)

D. The Main Experiments - The framework of the experiments, their stages, their characterizations and the populations of the research.

Each one of the next four chapters (E, F, G, H), which discuss the chapters of the course (see [15] the textbook), has the same structure with four stages as follows:

1. Description of the chapter - Review of the new concepts, the theorems and the exercises.
2. The Teaching - Mathematical and didactic analysis of the topics in the chapter with emphasis of new proofs, special examples or exercises et ceteras.
3. The Exercises - Analysis and review of the exercises including goals (according to Bloom [2], Tamir [10] and Chohen [3]).
4. The Learning Achievements - Analysis of the examinations, representation of the results, summary, conclusions and recommendations.

Some of the results in each of the four mentioned preceding chapters are presented here:

E. Graphs - Definitions, examples and basic concepts (see Table 1).

1. The concepts - Planar graph and isomorphic graphs were found out to be difficult concepts especially for 3 units students. However, the checking of "if given graphs are isomorphic or are planar" was found to be difficult even for 4 and 5 units students. Therefore we recommend to deal only with the concepts and simple examples.
2. Theorem A.2 and its proof are an example of our method of teaching and representation that doesn't exist in literature, and which is our contribution to the research. ([13], p. E-14)
3. The students understood well the concepts and the various types of graphs (Table 1).
4. The achievements of the 4 and 5 units students were better, especially in solving problems. ([13], p. E-14 and Table 2)

F. Paths and Circuits on Graphs (see Table 2).

Teaching of Graph Theory - Flow Chart of the Research

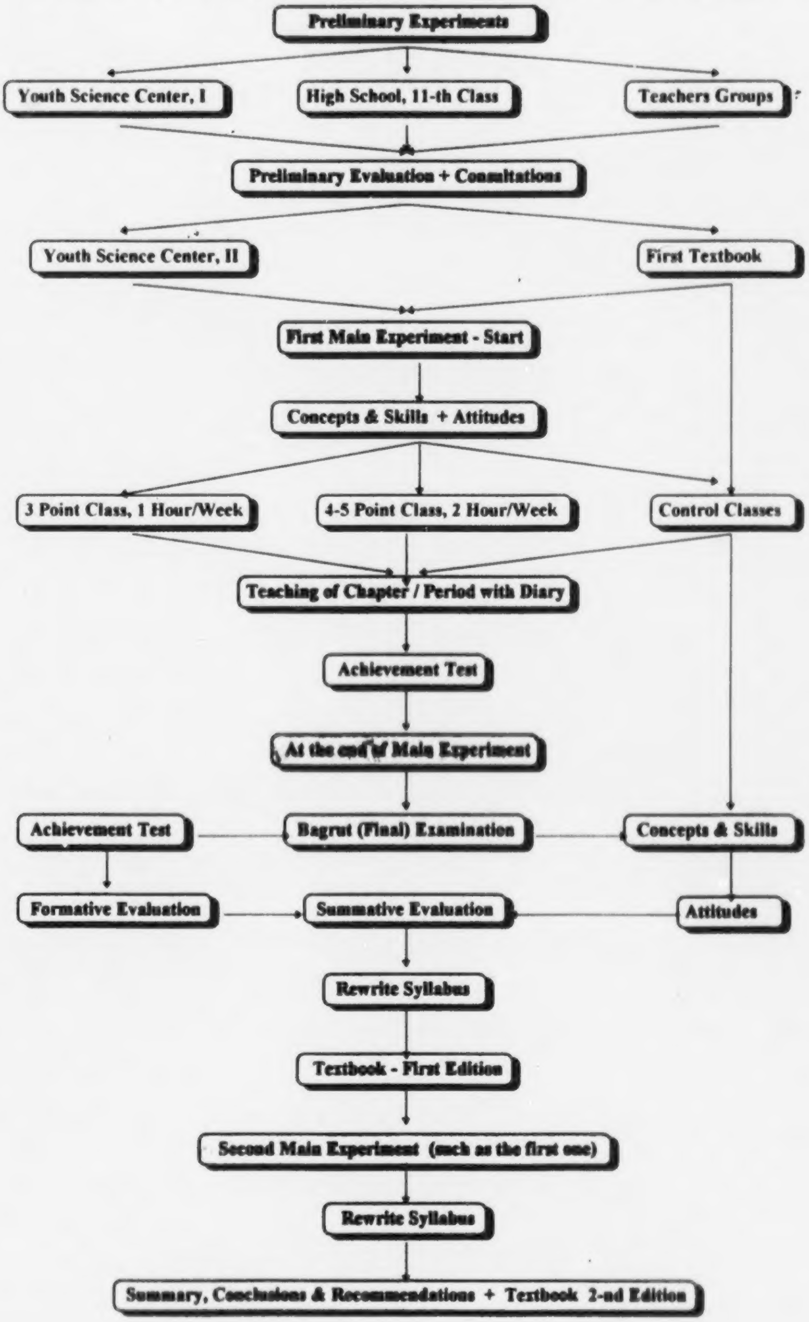

Preliminary Experiments
Youth Science Center, I
High School, 11-th Class
Teachers Groups
Preliminary Evaluation + Consultations
Youth Science Center, II
First Textbook
First Main Experiment - Start
Concepts & Skills + Attitudes
3 Point Class, 1 Hour/Week
4-5 Point Class, 2 Hour/Week
Control Classes
Teaching of Chapter / Period with Diary
Achievement Test
At the end of Main Experiment
Achievement Test
Bagrut (Final) Examination
Concepts & Skills
Formative Evaluation
Summative Evaluation
Attitudes
Rewrite Syllabus
Textbook - First Edition
Second Main Experiment (such as the first one)
Rewrite Syllabus
Summary, Conclusions & Recommendations + Textbook 2-nd Edition

Table 1

ACHIEVEMENTS

Chapter A: The Average Success Rate.

IV Problems	III True or False	II Types of Graphs	I Concepts	N	Population
78.7	80	90.6	98	14	A (4-5)
77.36	80.8	83	96.8	32	B (3)
86.5	89.5	94.5	86.66	20	C (4-5)
78.87	78.8	77.69	82.4	22	D (3)
77.75	42.91	79.2	89.2	40	E-F(4-5)
80·23	82·4	84·9	90·6	128	All Groups

Table 2

Chapter B: The Average Success Rate.

IV Problems	III True or False	II Theorems	I Concepts	N	Population
93.5	88.74	91.02	91·67	34	Poi.4-5-AC
86·4	78.09	86·65	94.44	54	Poi. 3 -BD
86·75	83.42	88·83	93.05	88	All -ABCD

This chapter contains 14 theorems which represented and proved in different ways than those of the academic textbooks that don't fit the high school students (see for example Theorem B.2 [13] p. F-12 and Peterson Graph p. F-50). The chapter includes also nearly 100 exercises. In fact we have needed more hours for teaching this chapter than we had conjectured. Below we list some of the results:

1. This chapter helped very much to understand the concepts of *models* and .*mathematical models.*

2. The students at all levels understood very well the topics *paths, connectivity* and *circuits* as well as the *Eulerian graphs.* But, some of the proofs on *optimal circuit* and on the *Hamiltonian graphs* (Dirac Theorem) were difficult even for the 5 units students.

3. More than 90% of the students understood well all the concepts (Table 2).

4. Some of the theorems were difficult (only 71% proved them completely).

5. The achievements of the 4 and 5 units students on solving problems were much better than those of the others. The same was found in the College classes.

G. Trees (see Table 3).

In this chapter the student meets with an important branch of graph theory, including nine theorems and algorithms. Teaching this chapter showed:

1. All the concepts were understood very well by all the students, therefore, we stopped checking concepts in the next chapter (Table 3).

2. In spite of the difference in the examinations, the achievements of the 4 and 5 units students were better than those of the 3 units students (Table 3).

3. Some of the proofs were difficult and unnecessary for the understanding of the chapter (Table 3). Therefore, we recommend omitting them from the course, especially for the 3 units students. For example, Cayley's Formula: The number marked spanning trees of Kn is nn-2.

Table 3

Chapter C Trees: The Average Success rate.

IV Problems	III True or False	II Theorems	I Concepts	N	Population
95.1	87.4	95.7	96.3	32	AC
83.8	87	72.1	87.6	54	BD

H. Networks, Games and Puzzles (see Table 4).

1. In this chapter we discussed two chapters (D,E of the textbook [15]) of the course. Because of the restriction on the outline of the PhD work, and because of the outline of the course the full discussion on these topics was moved to [14] Appendix 2.0.

2. Almost all of the chapter on networks is intended for 5 units students and only part of it is intended for 4 units students (Table 4).

3. In this chapter we used our idea of *stretch and constriction in graph*, which was very useful. ([13], p. H-2)

4. The main theorem in this chapter is the *min-max theorem*, which was proved and used (Table 4).

Table 4

Chapter D Networks: The Average Success Rate.

Population - AC , N=34

3 Max Flow & Min Cut	2 $c(H) \le v(f)$	1 Robin`s Theorem	Theorem Category
90	91.7	100	Full Answer
10	8.3	-	Partial Ans.
-	-	-	Wrong Answer
23.1	7.7	69.2	No Answer
100	100	100	Total

I. Comprehensive Evaluation (see Tables 5, 6, 7).

This chapter, the last one in our work, contains a general position in the three levels: Achievements on graph theory, attitudes in respect to graph theory and mathematical studies, and achievements in examination on concepts and skills in mathematics. Some of the main results are:

1. The achievements on graph theory are close to those in other parts of mathematical studies (Table 5).
2. The achievements of the 5 units students were the best and those of the 3 units were the next (Table 5).
3. It is possible and preferable to teach graph theory at all levels of the high school, and even in the middle school.
4. The attitudes of the students in the experiments classes and in the criticism classes in respect to mathematics were similar (Table 7).
5. We found a clear modification on the attitudes, in any population, between levels especially when comparing the attitudes at the beginning of the year to its end (Table 7). The attitudes of the 4 and 5 units students were more positive (Tables 6 and 7).
6. There was more improvement at the end of the year compared with the beginning on attitudes in respect to graph theory than in respect to mathematics (Table 7).
7. There is influence of graph theory on mathematical studies (comparison between experiments classes and criticism classes) (Table 7).
8. The achievements in the "Test - Concepts and Skills in Mathematics" (see Appendix 2) at the end of the year were better (in all classes) than at the beginning of the year (Table 8).

Table 5

Global Achievements:
Bagrut (Finals) Examinations - The Average Success Rate
A Comparison between Graph Theory and Algebra/Math.

Algebra/Mathematics			Graph Theory			
Wrong Answer	Partial Answer	Full Answer	Wrong Answer	Partial Answer	Full Answer	
8	22	70		14	86	3 Poi
9	22	69		22	78	4 poi
	27	73		10	90	5 poi

9. Theimprovement in the experiments classes was more significant than that of the criticism classes (Table 8).

10. The achievements of the 4 and 5 units students were better than those of the 3 units students (Table 8).

Table 6

Attitudes of 4-5 points Pupils, 2nd Experiment, N=15

Negativ	Positi.	V. Pos.	No Ans.	Subject	Question
0	66.7	33.3	0	Algebra	Interest
13.3	46.7	33.3	6.7	Geometry	
13.3	53.3	33.3	0	Graph Theory	
33.3	46.7	20	0	Trigonometry	
20	73.3	6.7	0	Algebra	Difficulty
33.3	40	20	6.7	Geometry	
20	53.3	26.7	0	Graph Theory	
20	60	6.6	13.3	Trigonometry	
0	80	20	0	Algebra	Understanding
13.3	66.7	13.3	6.7	Geometry	
20	53.3	26.7	0	Graph Theory	
20	53.3	26.7	0	Trigonometry	
0	33.3	66.7	0	Algebra	Need Of The Subject
6.7	46.7	40	6.7	Geometry	
13.3	66.7	20	0	Graph Theory	
26.7	40	26.7	6.7	Trigonometry	
20	26.7	53.3	0	Algebra	Assistance To solve Problems of Other Areas
40	40	13.3	6.7	Geometry	
26.7	40	26.7	6.7	Graph Theory	
60	26.7	0	13.3	Trigonometry	

For examples of verbal atitudes of students see appendix 1. Appendices 2, 3 and 4 are only 3 examples of many tools we have developed and used in the research.

Table 7

Attitudes:Comparison of the Experiments Classes and Control Classes

MATHEMATICS			TRIGONOMETRY			GRAPH THEORY			GEOMETRY			ALGEBRA			N	Time of Che.	Lev. / Poi.	Population
NEG.	POS.	V.P.	NEG.	POS.	V.P.	NEG.	POS.	V.P.	NEG.	POS.	V.P.	NEG.	POS.	V.P.				
25	33	41	-	-	-	46	34	19	31	28	39	19	38	43	22	I	3	B
8	41	49	-	-	-	30	43	26	14	43	42	3	40	57	23	II	3	B
14	45	40	19	44	37	25	19	56	-	-	-	10	46	44	12	I	4-5	A
25	50	25	3 8	49	11	13	51	36	-	-	-	11	51	38	9	II	4-5	A
32	35	31	34	38	26	26	48	24	38	30	30	24	38	38	20	II	3	D
20	52	24	32	47	12	19	53	27	21	49	23	8	56	36	15	II	4-5	C
21	44	29	27	44	25	-	-	-	26	44	21	11	43	41	64	One	4-5	Con.X
25	35	36	-	-	-	-	-	31	31	32	20	39	40	41	18	I	3	Con.Y
18	36	20	33	44	-	-	-	23	45	28	29	13	42	43	21	II	3	Con.Y

V.P. - Very Positive. Pos. - Positive. Neg. - Negative

Con-X - Control Classes, Sapir College.

Con-Y - Control Classes, Kfar Maymon.

Table 8

Tab I-22, The Achievements on The Test of Skills And Concepts at Mathematics

IV. Solving Problems					III. Find Model For					II. Describe The Model					I. Determine The Concepts						Kind of Answer
Six.5 Peop. &Rec.	.4 Divi. 1400	.3 Numb. Prob.	.2 Bicy. Rider	.1 Area Pro.	.5 S.V. T.	.4 Doule Di.Nu	.3 Mult. of 5	.1 Rec. Area	.1 Even Num.	* .5 $x=(-b\pm\sqrt{\delta})/2a$	.4 $2r\pi$	.3 $a^2+b^2=c^2$	.2 3n	.1 2n+1	.6 Mat. Mod.	.5 Graph Func.	.4 Mat. Pro.	.3 Pra. Pro.	.2 Mod-el	.1 Pha-rase	
3-Points, First Time, (At The Beginning Of The Year)																					
-	9.7	22.6	41.9	9.7	90.3	9.7	6.5	61.2	12.9	61.3	29.0	31.5	-	6.5	6.5	25.8	16.1	6.5	12.9	16.1	Full
-	-	9.7	19.4	12.9	-	9.7	32.2	6.5	9.7	-	6.5	3.2	-	-	3.2	-	6.5	6.5	-	3.2	Part.
-	6.5	19.3	29.0	35.5	6.5	19.4	9.7	22.6	41.9	3.2	25.8	22.6	19.4	12.9	3.2	3.2	-	-	-	-	Wrong
-	83.8	48.4	9.7	41.9	3.2	61.2	51.6	9.7	35.5	35.5	38.7	38.7	80.6	80.6	87.1	71.0	77.4	87.0	87.1	80.7	No Answ.
3 Points, 2nd Time, (At The End Of The Year)																					
21·1	21·1	89·4	100·0	10·5	94.7	78.9	26.3	89.4	57.9	89.5	10.5	89.4	21.1	42.1	36.8	52.6	36.8	47.4	42.1	15.8	Full
21.0	10.5	5.3	-	21.1	-	5.3	10.5	5.3	10.5	-	-	5.3	5.3	5.3	-	-	15·8	10·5	10·5	10·5	Part.
5.3	5.3	-	-	26.3	5.3	5.3	15.9	5.3	5.3	-	84.2	-	21.0	5.3	-	-	5.3	-	15.8	21.1	Wrong
52.6	63.1	5.3	-	42.1	-	10.5	47.3	-	26.3	10.5	5.3	5.3	52.6	47.3	63.2	47.4	42.1	42.1	31.6	52.6	No Answ.
4-5 Points, First Time (At The Beginning of Th Year)																					
-	-	47.3	73.7	42.1	73.6	47.4	21.1	78.9	57.9	100.0	42.1	89.5	63.2	78.9	-	63.1	52.6	57.9	10.5	47.4	Full
10.5	15.8	15.8	10.5	15.8	5.3	-	5.3	10.5	21.0	-	-	10.5	15.8	5.3	5.3	31.6	26.3	5.3	5.3	26.3	Part.
-	5.3	5.3	10.5	5.3	15.8	15.8	52.6	10.5	5.3	-	52.6	-	10.5	5.3	94.7	5.3	-	10.5	-	15.8	Wrong
89.5	78.9	31.6	5.3	36.8	5.3	36.8	21.0	-	15.8	-	5.3	-	10.5	10.5	-	-	21.1	26.3	84.2	10.5	No Answ.

$\delta=b^2-4ac$ *

Tab I-22, Contin.

IV. Solving Problems					III. Find Model For					II. Describe The Model					I. Determine The Concepts						Kind of Answer
Six.5 Peop. &Rec.	.4 Divi. 1400	.3 Numb. Prob.	.2 Bicy. Rider	.1 Area Pro.	.5 S.V. T.	.4 Doule Di.Nu	.3 Mult. of 5	.1 Rec. Area	.1 Even Num.	* .5 $x=(-b\pm\sqrt{\delta})/2a$	.4 $2r\pi$	.3 $a^2+b^2=c^2$	.2 3n	.1 2n+1	.6 Mat. Mod.	.5 Graph Func.	.4 Mat. Pro.	.3 Pra. Pro.	.2 Mod-el	.1 Phr-rase	
4-5 Points, 2nd Time (At The End of The Year)																					
71.4	21.4	64.3	85.7	57.2	92.8	64.3	64.3	85.7	100	100	64.3	100	71.4	100	50.0	85.7	92.8	100	50.0	42.8	Full
14.3	42.9	7.2	-	21.4	-	21.4	14.3	7.1	-	-	-	-	7.2	-	14.3	14.3	-	-	7.2	14.8	Part.
-	-	-	-	-	7.2	-	14.3	7.2	-	-	35.7	-	21.4	-	-	-	-	-	21.4	14.8	Wrong
14.3	35.7	28.5	14.3	21.4	-	14.3	7.1	-	-	-	-	-	-	-	35.7	-	7.2	-	21.4	28.6	No Answ.
Control Classes, First Time, (At The Beginning Of The Year)																					
-	-	-	6.3	-	37.5	-	-	37.5	37·5	62.6	18.7	68.8	-	6.3	6.3	31.3	50.0	31.3	6.3	37.5	Full
-	-	-	12.5	6.3	-	-	-	12.5	12.5	18.7	-	-	25.0	18.7	-	37·5	25·0	18·7	6·3	50·0	Part.
-	6.3	-	18.7	25.0	37.5	56.3	68.8	43.7	37.5	-	25.0	12.5	37.5	43.7	6.2	18.7	6.2	18.7	6.2	-	Wrong
100.0	93.7	100.0	62.5	68.7	25.0	43.7	31.2	6.3	12.5	18.7	56.3	18.7	37.5	31.3	87.5	12.5	18.8	31.3	81.2	12.5	No Answ.
Control Classes, 2nd Time, (At The End Of The Year)																					
-	-	28.6	33.3	4.8	66.7	23.8	9.5	52.4	38.1	66.7	38.1	80.9	19.0	23.8	-	9.5	14.3	33.3	9.5	33.3	Full
9.5	-	4.8	9.5	4.8	-	4.8	-	4.8	9:5	23.8	9.5	4.8	4.8	4.8	9.5	28.6	47.6	19.1	4.8	19.1	Part.
23.8	42.9	14.3	42.9	23.8	14.3	47.6	66.7	33.3	33.3	-	38.1	4.8	47.6	42.8	4.8	38.1	23.8	28.5	4.8	38.1	Wrong
66.7	57.1	52.3	14.3	66.6	19.0	23.8	23.8	9.5	19.1	9.5	14.3	9.5	28.6	28.6	85.7	23.8	14.3	19.1	80.9	9.5	No Answ.

$\delta=b^2-4ac$ *

TWO (BRIEF) EXAMPLES OF THE TEACHING METHODS

A. The Climb on the Infinite Ladder

As a mathematical tool for teaching the principle of mathematical induction.

1. Mathematical induction is used to establish with logical certainty the correctness of theorems and problems.
2. Many proofs in graph theory make use of math induction.
3. Many students of all levels (including university students) do not completely understand the principle of induction.
4. Many of the students are not convinced that the principle is valid. (The Assumption is exactly what is needed to prove.) Because of that, we developed the idea of the infinite ladder as follows:

The Problem

Suppose there is an infinite ladder which is lying on the ground with its top pointing to the sky.

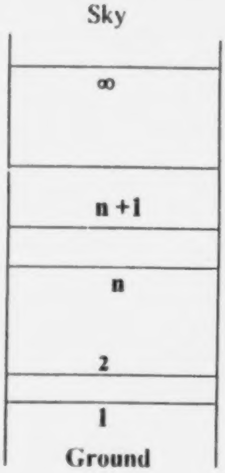
In addition, we know nothing about the height of the first step from the ground nor the distance between any two steps on the ladder.

Moreover, the distance between any two steps on the ladder is not uniform!

Can you climb the ladder endlessly?

Obviously a practical check isn't possible!

However, we can ensure that we can climb the ladder endlessly if:



a. It is possible to climb the first step.

b. It is always possible to climb from the n- step to the next one, n + 1 step, for any integer n .

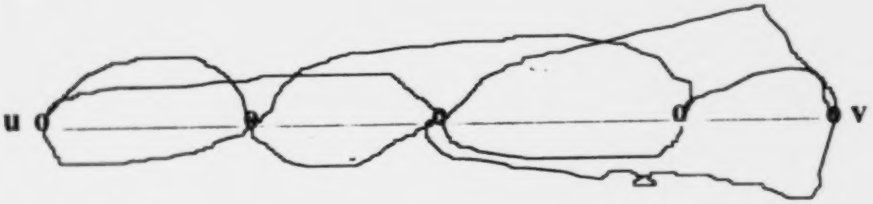
The analogy of this idea to the principle of *induction* is clear. Students, of all levels, understood it easily.

This demonstration was found to be a very useful tool to help students, of all levels, to understand the principle of mathematical induction.

B. The idea of constriction and stretch in graphs for finding the shortest path between two vertices in graphs.

Suppose that the edges of a given graph are plastic but not elastic and suppose that the graph was constricted. According to the Euclid's Theorem: "The shortest distance between two points is the straight line."

Let us pick up two vertices u,v of the given graph and stretch them until we get a straight line.



This is the shortest path between u and v because all the other paths are longer. With this idea in mind let us describe the idea of constriction and stretching of graphs, for finding the shortest path in a graph. (for more details see [14])

IV SUMMARY AND RECOMMENDATIONS

Our research showed that the main goal of the research—developing and writing a syllabus on graph theory for high school—was achieved. Therefore we recommend the following:

1. To teach graph theory within mathematical studies in high school in all levels.
2. In 3 units level, to teach only the first three chapters of the textbook.
3. To prepare a guidebook for the teacher including mathematical and didactic instructions, as well as more examples and more solved problems.
4. To integrate suitable topics of graph theory as enrichment chapters for the middle school, and even for elementary school.
5. To train teachers in colleges or by schooling to teach graph theory, at all levels.

APPENDIX 1

Examples of Verbal Attitudes of Students Towards Mathematics in General, and Graph Theory in Particular

Is the subject interesting?
3-pt Students, At the Beginning of the Year:
- No, whoever created math didn't have me in mind.
- Yes, it exercises logic and thinking processes.
- Very, I think it's very interesting, even though it's difficult for me.

3-pt Students, At the End of the Year:

- No, it doesn't advance me in further studies, so I would give it up.
- Very, difficult world problems can be solved in a graphical way.
- Very, it's a new way of mathematical thinking that I was hardly aware of.

4-pt Students, At the Beginning of the Year:

- Very, because I can apply this theory to my everyday life.

4-pt Students, At the End of the Year:

- Yes, because I really like its nature - the proofs and the logic. It's a bit like geometry
- Yes, sometimes you can use graphs for real things in life, and it's very interesting to see that there's "lawfulness" (=rules) for these things.

Should graph theory be taught?

3-pt Students, At the Beginning of the Year:

- No, what will I do with it?

3-pt Students, At the End of the Year:

- Yes, for enjoyment.
- Definitely, interest, challenge and success bring satisfaction.

4-5 pt Students, At the End of the Year:

- No, it's worthless, a waste of time.
- Yes, it develops students' thinking
- Yes, because sometimes it develops your intelligence

What's your opinion on mathematics in general, and graph theory, in particular?

3-pt Students

- The topic of graph theory isn't as monstrous as I pictured it at the beginning of the year. It's possible to understand it if you want to because it's only based on a few principles. It's worth continuing to teach it, even though not too many people know what it is.
- Graph theory is a new subject that seemed horrifying at the beginning, but after, most of the students got used to it, and found out it wasn't so bad.
- Graph theory should be practiced more, so that the material will be understood etter, like algebra and statistics.

4-5 pt. Students

- Graphs are a good subject, that can help us in many areas, like roads. And even for a lot of things that I wanted to know, but I didn't know, and today I know a lot.
- Math is usually a very interesting subject that helps us in the far future, but it is also irritating. Graph theory is a very interesting subject, that helps in many areas in the present and the future, but some things in it are very difficult.
- I think graph theory is like a new language; until you master it you have no interest in it.

APPENDIX 2

Test - Concepts and Skills in Mathematics

1. Determine the following concepts.

   a) phrase
   b) model
   c) practical problem
   d) mathematical problem
   e) function graph
   f) mathematical problem

2. What do the following formulas express?
   a) 2n+1 from n integer
   b) 3n from n integer
   c) $a^2 + b^2 = c^2$ (a, b, c -- the sides of an orthogonal triangle)
   d) $2\pi r$ (r -- the radius of a circle)
   e) $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

3. Find a formula that expresses:
   a) An even number.
   b) The area of a rectangle.
   c) A number which is divisible by 5, with remainder of 1.
   d) A two-digit number.
   e) The relationship between time, velocity and distance.

4. Solve the following problems:
   a) A square and a rectangle have the same perimeter.
   The width of the rectangle is 1/3 of its length.
   The area of the square is greater by 4, than the area of the rectangle.
   Find: 1. The length of the sides of the square.
   2. The area of the square.

   b) A bicycle was ridden at a constant speed for 4 hours. After that, it was ridden for another 3 hours, at a speed 5 km/h greater than before. In all, 51 km were traveled. What was the initial speed?

   c) The first digit of a two-digit number is greater by 2 than its tenth digit. If we will exchange the places of the digits in the number we will get a smaller number by 27 than the given number. Find the given number.

   d) How many different divisors does the number 1400 have?

   e) Show that in any group of six people there are always at least 3 people who know each other, or at least 3 people who don't know each other at all.

Good Luck !

APPENDIX 3

Diary of Teaching - Structured Observation
Diary of the lesson on Graph Theory

Date:___/___/___, Class________, 1 [] 2 [] lessons.

The subject of the lesson____________________________________

A. From the point of view of the pupil
   1. Understanding of the material (Theorems, Exercises..) in the class. Notation of special difficulties, flashes of understanding, etc.

   ____

   ____

2. The atmosphere in the class.
   Interest level:
   Workload:
   Level of difficulty:
   Comments of pupils:

B. From the point of view of the teacher - didactical aspects

1. Notation of difficulties in explaining the subject, specific theorem, or exercise

2. Is it preferable to expand, to omit, or to leave the subject as it is? Should an exercise be added? Are explanations for the exercises necessary or unnecessary? Is there a need to add solved examples? etc.

C. Discussions, Interviews and General Comments

APPENDIX 4

Test - Graph Theory, Chapter B
Path and Circuits In Graphs
(4-5 points, Kfar Maymon)

A. Determine the following concepts:
1. Connected graph.
2. Simple path.
3. Optimal Circuit.
4. Semi-Hamiltonian graph.
5. Eulerian graph.
6 .Weighted graph.

B. Prove 2 of the 3 following Theorems:
1. A complete bipartite graph - $K_{m,n}$ is Hamiltonian graph if and only if (iff) $m = n \geq 2$.
2. If a graph G is bipartite, hen G doesn't contain any odd circuit.
   Give the opposite theorem. Is it true? Give a reason.
3. If not all the vertices of a given graph G are of even degree, then it is possible to draw G with number of lineswhich equal to half of the number of the vertices of odd degree in G.

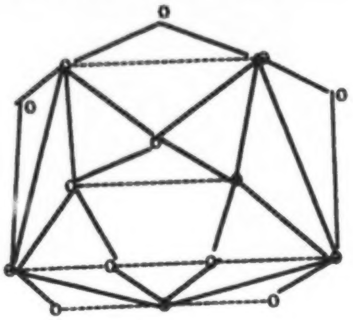
C. Determine "True" or "False" for each of the following Phrases.
If "true" prove it. If "false" give a contrary example.

4. Every graph which contain an odd circuit isn't bipartite graph.
5. Every graph which contain an cut-edge contains at least two cut-vertex.
6. Every Eulerian graph is Hamiltonian graph and vise versa.

7. A graph G which all its vertices are of exactly degree 2 contains one and only one circuit which contain all G.

D. Problems: Give a full proof to each of the following problems.

9. Prove: If a simple graph G with n vertices has $(n-1)(n-2)/2$ edges, then G is connected graph.
10. Eleven friends had a tour. Every night they had supper together around a round table so that every evening each one sat between two friends which he didn't sat between them before. How many days the tour was?
    Name the theorems you use at your proof.
11. Check the present graph if it is:
    Prove any answer you give!
    a. Is G Eulerian graph?
    b. Is G Semi-Eulerian?
    c. How many lines needed to draw G?
    d. Is G Hamiltonian graph?
    e. Is G Semi-Hamiltonian graph?



Good Luck!

REFERENCES

[1] Altshuler A. Graphs Polytops and Maps (Heb.), Israel: The Amos De-Shalit Science Teaching Center, 1972.

[2] Bloom B.S. Taxonomy of Educational Objectives. New York: McKay and Co., 1956.

[3] Chohen A. Development of thinking of the pupil in the class (Heb.). Jerusalem: Noam, 1991.

[4] Committee on Undergraduate Program in Mathematics. Applied Mathematics in Undergraduate Curriculum. Berkeley, California: 1972.

[5] Fujii J. N. Puzzles and Graphs. Reston, VA: The National Council of Teachers of Mathematics, 1973.

[6] Harary F. Graph Theory. Boston, MA: Addison Wesley, 1969.

[7] Levy A. (ed., Translated into Heb.) Readings in Curriculum Evaluation. Tel-Aviv: School of Education, Tel-Aviv University, 4th ed., 1980.

[8] Ore Oystein. Graphs and their Uses. Mathematical Association of America, Yale University, 1963.

[9] Robins H. and R. Curant. What is Mathematics. Oxford University Press, 1943, p. 235-271.

[10] Tamir P. Curriculum Implementation and its Relations with Curriculum Development (into Proceedings of the "Bat-Sheva" Conference). Jerusalem: The Amos de-Shalit Science Teaching Center, Hebrew University, 1978.

[11] Tamir P. and Mayer A.M. (ed.). Science Teaching in Israel Origins, Development and Achievements. Jerusalem: Science Teaching Center, Hebrew University, 1984.

[12] Temperly, H.N.V. Graph Theory and Applications. New York: John Wiley, 1981.

[13] Turgman A. Graph Theory for High School—Development Writing Experimentation and Evaluation (PhD Thesis), Hebrew University of Jerusalem, Israel, 1996.

[14] Turgman A. Graph Theory for High School —Development writing experimentaion and Evaluation—Appendices (PhD Thesis), Hebrew University of Jerusalem, Israel, 1996.

[15] Turgman A. Graph Theory (textbook of the Course), The Amos de-Shalit Science Teaching Centre, Hebrew University of Jerusalem (2nd edition), 1996.

[16] Turgman A. Graph Theory, Akademon press, Hebrew University of Jerusalem (3rd edition), 1997.

APPENDICES

APPENDIX A

WORKING GROUPS AT EACH ANNUAL MEETING

1977 Queen's University, Kingston, Ontario
  Teacher Education programmes
  Undergraduate mathematics programmes and prospective teachers
  Research and mathematics education
  Learning and teaching mathematics

1978 Queen's University, Kingston, Ontario
  Mathematics courses for prospective elementary teachers
  Mathematization
  Research in mathematics education

1979 Queen's University, Kingston, Ontario
  Ratio and proportion: a study of a mathematical concept
  Minicalculators in the mathematics classroom
  Is there a mathematical method?
  Topics suitable for mathematics courses for elementary teachers

1980 Université Laval, Québec, Québec
  The teaching of calculus and analysis
  Applications of mathematics for high school students
  Geometry in the elementary and junior high school curriculum
  The diagnosis and remediation of common mathematical errors

1981 University of Alberta, Edmonton, Alberta
  Research and the classroom
  Computer education for teachers
  Issues in the teaching of calculus
  Revitalising mathematics in teacher education courses

1982 Queen's University, Kingston, Ontario
  The influence of computer science on undergraduate mathematics education
  Applications of research in mathematics education to teacher training programmes
  Problem solving in the curriculum

1983 University of British Columbia, Vancouver, British Columbia
  Developing statistical thinking
  Training in diagnosis and remediation of teachers
  Mathematics and language
  The influence of computer science on the mathematics curriculum

1984 University of Waterloo, Waterloo, Ontario
  Logo and the mathematics curriculum
  The impact of research and technology on school algebra
  Epistemology and mathematics
  Visual thinking in mathematics

1985 Université Laval, Québec, Québec
- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 Memorial University of Newfoundland, St, John's, Newfoundland
- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 Queen's University, Kingston, Ontario
- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 University of Manitoba, Winnipeg, Manitoba
- Teacher education: what could it be
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 Brock University, St. Catharines, Ontario
- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 Simon Fraser University, Vancouver, British Columbia
- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 University of New Brunswick, Fredericton, New Brunswick
- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 ICME-7, Université Laval, Québec, Québec

1993 York University, Toronto, Ontario
- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 University of Regina, Regina, Saskatchewan
Theories of mathematics education
Preservice mathematics teachers as pruposeful learners: issues of enculturation
Popularizing mathematics

1995 University of Western Ontario, London, Ontario
Anatomy and authority in the design and conduct of learning activity
Expanding the conversation: trying to talk about what our theories don't talk about
Factors affecting the transition from high school to university mathematics
Geometric proofs and knowledge without axioms

1996 Mount Saint Vincent University, Halifax, Nova Scotia
Teacher education: challenges, opportunities and innovations
Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
What is dynamic algebra?
The role of proof in post-secondary education

1997 Lakehead University, Thunder Bay, Ontario
Awareness and Expression of Generality in Teaching Mathematics
Communicating Mathematics
The Crisis in School Mathematics Content

APPENDIX B

PLENARY LECTURES

1977	A.J. Coleman	The objectives of mathematics education
	C. Gaulin	Innovations in teacher education programmes
	T.E. Kieren	The state of research in mathematics education
1978	G.R. Rising	The mathematician's contribution to curriculum development
	A.I. Weinzweig	The mathematician's contribution to pedagogy
1979	J. Agassi	The Lakatosian revolution*
	J.A. Easley	Formal and informal research methods and the cultural status of school mathematics*
1980	C. Cattegno	Reflections on forty years of thinking about the teaching of mathematics
	D. Hawkins	Understanding understanding mathematics
1981	K. Iverson	Mathematics and computers
	J. Kilpatrick	The reasonable effectiveness of research in mathematics education*
1982	P.J. Davis	Towards a philosophy of compuation*
	G. Vergnaud	Cognitive and developmental psychology and research in mathematics education*
1983	S.I. Brown	The nature of problem generation and the mathematics curriculum
	P.J. Hilton	The nature of mathematics today and implications for mathematics teaching*
1984	A.J. Bishop	The social construction of meaning: a significant development for mathematics education?*
	L. Henkin	Linguistic aspects of mathematics and mathematics instruction
1985	H. Bauersfeld	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. Pollak	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. Finney	Professional applications of undergraduate mathematics
	A.H. Schoenfeld	Confessions of an accidental theorist*
1987	P. Nesher	Formulating instructional theory: the role of students' misconceptions*
	H.S. Wilf	The calculator with a college education
1988	C. Keitel	Mathematics education and technology*
	L.A. Steen	All one system

1989	N. Balacheff	Teaching mathematical proof: the relevance and complexity of a social approach
	D. Schattsneider	Geometry is alive and well
1990	U. D'Ambrosio	Values in mathematics education*
	A. Sierpinska	On understanding mathematics
1991	J .J. Kaput	Mathematics and technology: multiple visions of multiple futures
	C. Laborde	Approches théoriques et méthodologiques des recherches Francaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. Joseph	What is a square root? A study of geometrical representation in different mathematical traditions
	J Confrey	Forging a revised theory of intellectual development Piaget, Vygotsky and beyond*
1994	A. Sfard	Understanding = Doing + Seeing ?
	K. Devlin	Mathematics for the twent-first century
1995	M. Artigue	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. Millett	Teaching and making certain it counts
1996	C. Hoyles	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. Henderson	Alive mathematical reasoning
1997	R. Borassi	What does it really mean to teach mathematics through inquiry?
	P. Taylor	The high school math curriculum
	T. Kieren	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM

*These lectures, some in a revised form, were subsequently published in the journal *For the Learning of Mathematics*.

APPENDIX C

PROCEEDINGS OF ANNUAL MEETINGS OF CMESG/GCEDM

Past proceedings of the Study Group have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	Not yet assigned*

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

*These *Proceedings* have been submitted to ERIC.

Appendix D

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